DOCUMENT 122-07

TELEMETRY GROUP

UNCERTAINTY ANALYSIS PRINCIPLES AND METHODS

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UNCERTAINTY ANALYSIS PRINCIPLES AND METHODS

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PREFACE

The Telemetry Group of the Range Commanders Council prepared this document to

define and catalog standard mathematical concepts and methods used to estimate measurement uncertainty in test instrumentation systems. This document addresses detailed analysis procedures for use in identifying measurement process errors and in estimating their uncertainty. The document is structured so that specific measurement methodologies of interest can be easily accessed by the reader.

Depending on the reader's knowledge and background, he/she may want to review the two "up-front"sections immediately following this preface. The first section is an overview of this document offering a description of analysis methods discussed. The second section contains a review of key uncertainty analysis concepts.

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OVERVIEW OF THIS DOCUMENT

This document has been prepared to define and catalog standard mathematical concepts and methods used to estimate measurement uncertainty in test instrumentation systems.

The analysis methods outlined in this document provide a comprehensive approach to estimating measurement uncertainty. Basic guidelines are presented for estimating the uncertainty for the following measurement processes:

- Direct Measurements – The value of the quantity or subject parameter is obtained directly by measurement and is not determined indirectly by computing its value from the measurement of other variables or quantities.

- Multivariate Measurements – The value of the quantity or subject parameter is based on measurements of more than one attribute or quantity.

- Measurement Systems – The value of the quantity or subject parameter is measured with a system comprised of component modules arranged in series.

The structured, step-by-step analysis procedures described address the important aspects of identifying measurement process errors and estimating their uncertainty. Advanced topics cover uncertainty growth over time and the refinement of uncertainty estimates using Bayesian methods.

Approach

The uncertainty that is determined and reported for a particular measurement should be the most realistic estimate possible. In this regard, the person tasked with conducting an uncertainty analysis must be knowledgeable about the measurement process under investigation. To facilitate this endeavor, the measurement process should be described in written format. This write-up should clearly specify the measurement equipment used, the environmental conditions during measurement, and the procedure used to obtain the measurement.

The approach presented in this document provides straightforward and easy-to-understand principles of measurement uncertainty analysis for direct and multivariate measurements and measurement systems.[1] Concepts and methods are consistent with those found in the ISO Guide to the Expression of Uncertainty in Measurement (ISO GUM).[2] Advanced measurement uncertainty analysis topics that extend these methods and concepts are also presented.

UncertaintyAnalyzer, developed by the Integrated Sciences Group (ISG), mentioned throughout this document is a software tool for implementing and demonstrating the methods and calculations described herein. Protocols, developed and implemented by ISG, to validate the UncertaintyAnalyzer program are discussed and examples are presented that compare UncertaintyAnalyzer calculations to values obtained by hand calculations and from Excel spreadsheets.

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[1] Concepts and methods presented herein are taken from material developed by Integrated Sciences Group (see References).

[2] *U.S. Guide to the Expression of Uncertainty in Measurement*, ANSI/NCSL-Z540-2-1997.

Other Methods

Numerous measurement uncertainty analysis standards and guides have been published over the past twenty years or so. Many of the uncertainty analysis references developed prior to the ISO GUM are based on techniques developed by Dr. Robert B. Abernathy and colleagues. Examples of uncertainty analysis standards and other published material commonly used in the U.S. engineering community are:

- Test Uncertainty, ASME PTC 19.1-1998 (reaffirmed 2004).
- Measurement Uncertainty for Fluid Flow in Closed Conduits, ANSI/ASME MFC-2M-1983 (reaffirmed 2001).
- Assessment of Wind Tunnel Data Uncertainty, AIAA Standard S-071-1995.
- Dieck, R.H.: Measurement Uncertainty Methods and Applications, 3rd Edition, ISA 2002.
- Coleman, H. W. and Steele, W. G.: Experimentation and Uncertainty Analysis for Engineers, 2nd Edition, John Wiley & Sons, 1999.

Although many of these references have been updated or reaffirmed in recent years, the uncertainty analysis methods that they espouse are distinctly different from those presented in the ISO GUM. Key differences are summarized below to illustrate how the methods and concepts presented in this document supplant techniques that are not based on the properties of measurement error and the statistical nature of measurement uncertainty.

1. **Measurement Error.** Prior to the ISO GUM, measurement errors were categorized as either random or systematic.[3] In this context, random error is defined as the portion of the total measurement error that varies as a result of repeat measurements of a quantity. Systematic error is defined as the portion of the total measurement error that remains constant in repeat measurements of a quantity.

   The ISO GUM refers only to errors that can occur in a given measurement process and does not differentiate them as random or systematic. Measurement process errors can result from repeat measurements, operator bias, instrument parameter bias, resolution, environmental conditions, or other sources. The key is to identify all sources of error for a given measurement process. Additionally, each measurement error is considered to be a random variable that can be characterized by a statistical distribution.

2. **Uncertainty.** Uncertainty due to random error is computed from the standard deviation, SX, of a sample of data

$$S = \sqrt{\sum_{k=1}^{N} \frac{\left(X_k - \overline{X}\right)^2}{N-1}}$$

   where N is the sample size, X_k is the kth measured value, and $\overline{X}$ is the sample mean value.

   If the mean value is reported, then the standard deviation in the mean value is given by

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[3] The terms random and precision are often used interchangeably, as are the terms systematic and bias.

$$S_{\bar{x}} = \frac{S_x}{\sqrt{N}} .$$

Abernathy originally proposed that the uncertainty due to random error be computed by multiplying the sample standard deviation by the student's t-statistic with 95% confidence level.

$$U_{ran} = t_{95}S_x \quad \textbf{or} \quad U_{ran} = t_{95}S_{\bar{x}}$$

Since the publication of the ISO GUM, some uncertainty analysis references have defined random uncertainty as being equal to the standard deviation.

$$U_{ran} = S_x \quad \textbf{or} \quad U_{ran} = S_{\bar{x}}$$

In all references, uncertainty due to systematic error, U_{bias}, is based on past experience, manufacturer specifications, or other information. In most cases, this uncertainty estimate is assumed to be roughly equivalent to a 95% confidence interval or limits for a systematic error that is normally distributed with infinite degrees of freedom. For example, if $\pm B$ represents the 95% confidence limits for a normally distributed systematic error, then

$$U_{bias} = \pm B .$$

The ISO GUM supplants systematic and random uncertainties with standard uncertainty,[4] which is a statistical quantity equivalent to the standard deviation of the error distribution. In this regard, uncertainty is not considered to be a $\pm$ limit or interval. The standard uncertainty of a measurement error is determined from Type A or Type B estimates. Type A uncertainty estimates are obtained by the statistical analysis of a sample of data. Type B uncertainty estimates are obtained by heuristic means.

With a basic understanding of error distributions and their statistics, the uncertainty, u_x, in a measurement, $x = x_{true} + \varepsilon_x$, is the square root of the variance in the measurement error, ε_x.

$$u_x = \sqrt{\text{var}(x)} = \sqrt{\text{var}(x_{true} + \varepsilon_x)} = \sqrt{\text{var}(\varepsilon_x)}$$

Variance is a statistical quantity, defined as the mean square dispersion of the error distribution about its mean or mode value. Distribution variance provides a crucial link between measurement error and measurement uncertainty. To properly apply this method, the distribution selected to estimate uncertainty for a given error source should provide the most realistic statistical characteristics.

3. **Total Uncertainty.** For decades, the question of how to combine random and systematic uncertainties has been a major issue, often subject to heated debate. The view supported by many data analysts and engineers was to simply add the uncertainties linearly (ADD).

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[4] In this document, the terms standard uncertainty and uncertainty are used interchangeably.

$$U_{ADD} = \mp\left(B + t_{95}\frac{S}{\sqrt{N}}\right)$$

The view supported by statisticians and measurement science professionals was to combine them in root sum square (RSS).

$$U_{RSS} = \mp\sqrt{B + \left(t_{95}\frac{S}{\sqrt{N}}\right)^2}$$

In the late 1970s, a compromise was proposed in which either method could be used as long as the following constraints were met:

    a. The elemental random uncertainties and the elemental systematic uncertainties be combined separately.

    b. The total random uncertainty and total systematic uncertainty be reported separately.

    c. The method used to combine the total random and total systematic uncertainties be stated.

Ironically, it was also recommended that the RSS method be used to combine the elemental random uncertainties, S_i, and the elemental systematic uncertainties, B_i, as shown below.

$$S = \frac{1}{N}\left[\sum_{i=1}^{K} S_i^2\right]^{1/2} \qquad\qquad B = \left[\sum_{i=1}^{K} B_i^2\right]^{1/2}$$

Since the publication of the ISO GUM, most uncertainty analysis references now state that the total random and total systematic uncertainties be combined in RSS. In many instances, the student's t-statistic, t_{95}, is set equal to 2 and U_{RSS} is replaced by U_{95}.

$$U_{95} = \mp 2\sqrt{\left(\frac{B}{2}\right)^2 + \left(\frac{S}{\sqrt{N}}\right)^2}$$

U_{95} assumed to be equivalent to 95% confidence limits.

In the ISO GUM, the uncertainty in the value of an error is equal to the square root of the variance of the error distribution. Consequently, variance addition is used to combine uncertainties from different error sources.

To illustrate variance addition, let us consider the measurement of a quantity x that involves two error sources ε_1 and ε_2.

$$x = x_{true} + \varepsilon_1 + \varepsilon_2$$

The uncertainty in x is obtained from

$$u_x = \sqrt{\mathrm{var}(x_{true} + \varepsilon_1 + \varepsilon_2)} = \sqrt{\mathrm{var}(\varepsilon_1 + \varepsilon_2)}$$
$$= \sqrt{\mathrm{var}(\varepsilon_1) + \mathrm{var}(\varepsilon_2) + \mathrm{cov}(\varepsilon_1, \varepsilon_2)}$$

where the covariance term, $cov(\varepsilon_1, \varepsilon_2)$, is the expected value of the product of the deviations of ε_1 and ε_2 from their respective means. The covariance of two *independent* variables is zero. The covariance can be replaced with the correlation coefficient, $\rho_{1,2}$, which is defined as

$$\rho_{1,2} = \frac{\mathrm{cov}(\varepsilon_1, \varepsilon_2)}{u_1 u_2}$$

where

$$u_1 = \sqrt{\mathrm{var}(\varepsilon_1)} \qquad \text{and} \qquad u_2 = \sqrt{\mathrm{var}(\varepsilon_2)}.$$

Therefore, the uncertainty in x can be expressed as

$$u_x = \sqrt{u_1^2 + u_2^2 + 2\rho_{1,2} u_1 u_2}.$$

Since correlation coefficients range from minus one to plus one, this expression provides a more general and mathematically rigorous method for combining uncertainties. If, for example, $\rho_{1,2} = 0$ (i.e., statistically independent errors), then the uncertainties are combined using RSS. If $\rho_{1,2} = 1$, then the uncertainties are added. If $\rho_{1,2} = -1$, then the uncertainties are subtracted. Thus, variance addition

4. **Degrees of Freedom.** When uncertainties are combined, it is important to estimate the degrees of freedom for the total uncertainty. Prior to the ISO GUM, there was no way to estimate the degrees of freedom for uncertainties due to systematic error. Consequently, there was no way to compute the degrees of freedom for total uncertainty.

   Annex G of the ISO GUM provides a relationship for computing the degrees of freedom for a Type B uncertainty estimate

$$\nu \approx \frac{1}{2} \frac{u^2(x)}{\sigma^2[u(x)]} \approx \frac{1}{2} \left[\frac{\Delta u(x)}{u(x)} \right]^{-2}$$

where $\sigma^2[u(x)]$ is the variance in the uncertainty estimate, $u(x)$, and $\Delta u(x)$ is the relative uncertainty in the uncertainty estimate.[5]

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[5] This equation assumes that the underlying error distribution is normal.

Since the publication of the ISO GUM, a methodology for determining $\sigma^2[u(x)]$ and computing the degrees of freedom for Type B estimates has been developed by Dr. Howard T. Castrup.[6]

In the ISO GUM, the effective degrees of freedom, v_{eff}, for the total uncertainty, u_T, resulting from the combination of uncertainties u_i and associated degrees of freedom, v_i, for n error sources is estimated using the Welch-Satterthwaite formula

$$v_{eff} \approx \frac{u_{T*}^4}{\sum_{i}^{n} \frac{u_i^4}{v_i}}$$

where the total uncertainty u_{T*} is computed assuming no error source correlations.

5. **Confidence Limits.** Total uncertainty, u_T, and degrees of freedom, v_{eff}, can be used to establish the upper and lower limits that contain the true value (estimated by the mean value $\bar{x}$), with some specified confidence level, p.

Confidence limits are expressed as

$$\bar{x} - t_{\alpha/2, v_{eff}} u_T \leq true\,value \leq \bar{x} + t_{\alpha/2, v_{eff}} u_T$$

where $\alpha = 1 - p$ and the t-statistic, $t_{\alpha/2veff}$, is a function of both the degrees of freedom and the confidence level.

Prior to the ISO GUM, the total uncertainty U_{ADD}, U_{RSS} or U_{95}, was offered as type of confidence limit.

$$\bar{x} - U_{95} \leq true\,value \leq \bar{x} + U_{95}$$

In some respects, these limits are similar to the expanded uncertainty, ku, presented in the ISO GUM as an approximate confidence limit, in which the coverage factor, k, is used in place of the t-statistic.

$$\bar{x} - ku \leq true\,value \leq \bar{x} + ku$$

Unfortunately, the introduction of an expanded uncertainty has served to perpetuate confusion about what measurement uncertainty actually represents. The methods and concepts presented in this document are intended to provide necessary clarification about this and other uncertainty analysis issues.

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[6] See Dr. Castrup's works throughout the Reference section.

REVIEW OF KEY UNCERTAINTY ANALYSIS CONCEPTS

The following paragraphs provide a brief overview of key uncertainty analysis concepts and methods that are detailed in subsequent chapters. The general uncertainty analysis procedure consists of the following steps:

1. Define the Measurement Process
2. Identify the Error Sources and Distributions
3. Estimate Uncertainties
4. Combine Uncertainties
5. Report the Analysis Results

1. Define the Measurement Process

The first step in any uncertainty analysis is to identify the physical quantity whose value is estimated via measurement. This quantity may be a directly measured value or indirectly determined through the measurement of other variables. The former type of measurements are called "direct measurements," while the latter are called "multivariate measurements."

For multivariate measurements, it is important to develop an equation that defines the mathematical relationship between the quantity of interest and the measured variables.

$$S = f(x, y, z)$$

where

S = subject parameter or quantity of interest

f = mathematical function that relates S to measured quantities x, y, and z.

At this stage of the analysis, it is also important to briefly describe the test setup, measurement procedures, environmental conditions, instrument specifications and other relevant information that can help identify the measurement process errors.

2. Identify the Error Sources and Distributions

In any given measurement scenario, each measured quantity is a potential source of error. The basic error model for a measured quantity, x_{meas}, is

$$x_{meas} = x_{true} + \varepsilon_x$$

where ε_x is the measurement error.

Measurement process errors are the basic elements of uncertainty analysis. Once these fundamental error sources have been identified, we can begin to develop uncertainty estimates. Measurement errors most often encountered include, but are not limited to the following:

- Measurement Bias
- Random or Repeatability Error
- Resolution Error
- Digital Sampling Error

- Computation Error
- Operator Bias
- Environmental Factors Error
- Stress Response Error

Another important aspect of the uncertainty analysis process is that measurement errors can be characterized by statistical distributions. The statistical distribution for a given measurement error is a mathematical description that relates the frequency of occurrence of values with the values themselves.

In general, there are three error distributions that have been found to be relevant to most real world measurement applications: normal, lognormal, and Student's t. Measurement errors can also be characterized by other distributions such as the uniform, triangular, quadratic, cosine, exponential, and U-shaped, although they are rarely applicable.

The normal distribution should be applied as the default distribution, unless information to the contrary is available. The Student's t distribution is applied if the underlying distribution is normal, but the uncertainty estimate is obtained from a sample of measurements. The lognormal distribution should be applied if it is suspected that the distribution of the value of interest is skewed. When using the normal or lognormal distribution, some effort must be made to estimate a containment probability.

If 100% containment has been observed, then the following error distribution selection criteria are recommended:

- Apply the cosine distribution if the value of interest has been subjected to random usage or handling stress, and is assumed to possess a central tendency.

- Apply either the quadratic or half-cosine distribution, as appropriate, if it is suspected that values are more evenly distributed.

- The triangular distribution may be applicable, under certain circumstances, when dealing with parameters following testing or calibration. The triangular distribution may also be applicable for errors due to linear interpolation of tabulated data.

- Apply the U-shaped distribution if the value of interest is the amplitude of a sine wave incident on a plane with random phase.

- Apply the uniform distribution for resolution error due to digital readout. This distribution is also applicable for quantization error and RF phase angle error.

3. Estimate Uncertainties

As previously stated, all measurements are accompanied by error. Our lack of knowledge about the sign and magnitude of measurement error is called measurement uncertainty. This leads us to Axiom 1.

Axiom 1 - The uncertainty in a measured value is equal to the uncertainty in the measurement error.

Since errors can be described in such a way that their sign and magnitude have some definable probability of occurrence, we have Axiom 2.

Axiom 2 - Measurement errors follow statistical distributions.

With a basic understanding of error distributions and their statistics, we can estimate uncertainties. We begin with the statistical quantity called the **variance**, which is defined as the mean square dispersion of the distribution about its mean or mode value.

$$\text{var}(x) = \text{Mean Square Dispersion in } x$$

If a quantity x is a random variable representing a population of measurements, then the variance in x is just the variance in the error in x, which is expressed by the symbol ε_x.

$$\text{var}(x) = \text{var}(x_{\text{true}} + \varepsilon_x) = \text{var}(\varepsilon_x).$$

Finally, we have Axiom 3 to the crucial link between measurement error and measurement uncertainty.

Axiom 3 - The uncertainty, u, in a measurement is the square root of the variance in the measurement error.

Consequently, the uncertainty in the measured value x can be written as

$$u_x = \sqrt{\text{var}(x)} = \sqrt{\text{var}(\varepsilon_x)}$$

There are two approaches to estimating variance and uncertainty. Type A estimates involve data sampling and analysis. Type B estimates use engineering knowledge or recollected experience of measurement processes. The basic methods used to estimate Type A and Type B uncertainties are presented in Chapters 2 and 3, respectively.

4. Combine Uncertainties

Because the uncertainty in the measurement error is equal to the square root of the variance of the error distribution, we can use **variance addition** to combine uncertainties from different error sources.

For purposes of illustration, let us consider a quantity z that is obtained indirectly from the measurement of the quantities x and y via the linear function

$$z = ax + by$$

where the coefficients a and b are constants.

In this case, we are interested in the uncertainty in z in terms of the uncertainties in the measured quantities x and y. Additionally, measurement errors for x and y are composed of various process errors (e.g., random, bias, resolution, environmental, operator, etc.).

The variance of z can be expressed in terms of the variances of the individual variables, x and y

$$\text{var}(z) = \text{var}(ax + by) = a2\,\text{var}(x) + b^2\,\text{var}(y) + 2ab\,\text{cov}(x,y)$$

where the last term is the **covariance** between x and y.

The covariance can be replaced by the **correlation coefficient**, $\rho_{x,y}$, which is defined as

$$\rho_{x,y} = \frac{\text{cov}(x, y)}{u_x u_y}.$$

With this relationship and Axiom 3, we can express the variance of z as

$$u_z^2 = a^2 u_x^2 + b^2 u_y^2 + 2ab\rho_{x,y}u_x u_y.$$

The above equation can be generalized to cases where there are n measured quantities $x_1, x_2, \ldots x_n$.

$$\text{var}(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i^2 \, \text{var}(x_i) + 2\sum_{i=1}^{n} \sum_{j>1}^{n} a_i a_j \rho_{ij} u_i u_j$$

$$= \sum_{i=1}^{n} a_i^2 u_i^2 + 2\sum_{i=1}^{n} \sum_{j>1}^{n} a_i a_j \rho_{ij} u_i u_j$$

With variance addition, we have a logical approach for combining uncertainties that accounts for correlations between error sources. To assess the impact of the correlated errors on combined uncertainty, let us consider the measurement of a quantity x that involves two error sources ε_1 and ε_2.

$$x = z_{true} + \varepsilon_1 + \varepsilon_2$$

From Axioms 1 and 3 and variance addition, the uncertainty in x is obtained from

$$u_x = \sqrt{\text{var}(x_{true} + \varepsilon_1 + \varepsilon_2)} = \sqrt{\text{var}(\varepsilon_1 + \varepsilon_2)} = \sqrt{u_1^2 + u_2^2 + \rho_{1,2}u_1 u_2} \; .$$

The correlation coefficient, $\rho_{1,2}$, for two error sources can range in value from -1 to +1. If the two error sources are statistically independent, then $\rho_{1,2} = 0$ and

$$u_x = \sqrt{u_1^2 + u_2^2} \; .$$

Therefore, the uncertainties of statistically independent error sources are combined in a root-sum-square (RSS) manner. Conversely, if the two error sources are strongly correlated then $\rho_{1,2} = 1$ or -1. If $\rho_{1,2} = 1$, then

$$u_x = \sqrt{u_1^2 + u_2^2 + 2u_1 u_2} = \sqrt{(u_1 - u_2)^2} = |u_1 - u_2|,$$

and the uncertainties are combined linearly. If two error sources are strongly correlated and compensate for one another, then $\rho_{1,2} = -1$ and

$$u_x = \sqrt{u_1^2 + u_2^2 - 2u_1 u_2} = \sqrt{(u_1 - u_2)^2} = |u_1 - u_2|.$$

Therefore, the combined uncertainty is the absolute value of the difference between the individual uncertainties.

Measurement process errors for a given quantity aren't typically correlated. Consequently, it is safe to assume that there are no correlations between the following measurement process errors:

- Random Error and Parameter Bias ($\rho_{ran,bias} = 0$)
- Random Error and Operator Bias ($\rho_{ran,oper} = 0$)
- Parameter Bias and Resolution Error ($\rho_{bias,res} = 0$)

- Parameter Bias and Operator Bias ($\rho_{bias,oper} = 0$)
- Operator Bias and Environmental Factors Error ($\rho_{oper,env} = 0$)
- Resolution Error and Environmental Factors Error ($\rho_{res, env} = 0$)
- Digital Resolution Error and Operator Bias ($\rho_{dres,oper} = 0$)

In some instances, the measurement process errors for different quantities may be correlated. Accounting for cross-correlations is discussed in Chapter 6.

When uncertainties are combined, it is important to estimate the degrees of freedom for the total uncertainty. The effective degrees of freedom, v_{eff}, for the total uncertainty, u_T, resulting from the combination of uncertainties u_i and associated degrees of freedom, v_i, for n error sources is estimated using the Welch-Satterthwaite formula.[7]

$$v_{eff} = \frac{u_{T*}^4}{\sum_i^n \dfrac{u_i^4}{v_i}}$$

5. Report the Analysis Results

When reporting the results of an uncertainty analysis, the following information should be included:

1. The estimated value of the quantity of interest and its combined uncertainty and degrees of freedom.
2. The mathematical relationship between the quantity of interest and the measured components.
3. The value of each measurement component and its combined uncertainty and degrees of freedom.
4. A list of the measurement process uncertainties and associated degrees of freedom for each component, along with a description of how they were estimated.
5. A list of applicable correlation coefficients, including any cross-correlations between component uncertainties.

It is also a good practice to provide a brief description of the measurement process, including the procedures and instrumentation used, and additional data, tables and plots that help clarify the analysis result

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[7] This formula is based on the assumption that there are no correlations between error source uncertainties.

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ACRONYMS

ADC	Analog to Digital Converter
AOA	Angle of Attack
AOP	Average-over-Period
BOP	Beginning-of-Period
DAC	Digital to Analog Converter
DMM	Digital Multimeter
EOP	End-of-Period
FS	Full Scale
FSI	Full Scale Input
FSO	Full Scale Output
ISG	Integrated Sciences Group
ISO	International Standard for Organization
ISO GUM	ISO Guide to the Expression of Uncertainty in Measurement
LSBF	Least Squares Best Fit
MTE	Measuring and Test Equipment
OOT	Out of Tolerance
PDF	Probability Density Function
RF	Radio Frequency
RSS	Root-sum-square method of combining values
	Residual sum of squares in regression analysis
SU	Subject Unit
TME	Test and Measurement Equipment
UUT	Unit Under Test
VBA	Visual Basic for Applications
Vdc or VDC	Volts Direct Current

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CHAPTER 1

BASIC CONCEPTS AND METHODS

1.0 General

This chapter describes the basic concepts and methods used to estimate measurement uncertainty.[8] The general uncertainty analysis procedure consists of the following steps:

1. Define the Measurement Process
2. Identify the Error Sources and Distributions
3. Estimate Uncertainties
4. Combine Uncertainties
5. Report the Analysis Results

The following sections discuss these analysis steps in detail. A discussion on using uncertainty estimates to compute confidence intervals and expanded uncertainties is also included.

1.1 Definitions

1.1.1 <u>Combined Uncertainty</u>. The uncertainty in the total error of a value of interest.

1.1.2 <u>Computation Error</u>. The error in a quantity obtained by computation. Normally due to machine round-off error in values obtained by iteration. Sometimes applied to errors in tabulated physical constants.

1.1.3 <u>Confidence Level</u>. The probability that a set of error limits or containment limits will contain errors for a given error source.

1.1.4 <u>Confidence Limits.</u> Limits that bound errors for a given error source with a specified probability or "confidence."

1.1.5 <u>Containment Limits</u>. Limits that are specified to contain either a parameter value, deviations from the nominal parameter value, or errors in the measurement of the parameter value.

1.1.6 <u>Containment Probability</u>. The probability that a parameter value or errors in the measurement of this value will lie within specified containment limits.

1.1.7 <u>Correlation Analysis</u>. An analysis that determines the extent to which two error sources influence one another. Typically the analysis is based on ordered pairs of values of the two error source variables.

1.1.8 <u>Correlation Coefficient</u>. A measure of the extent to which two error sources are linearly related. A function of the covariance between the two error sources. Correlation coefficients range from minus one to plus one. A positive correlation coefficient applies when increases in one source are accompanied by increases in the other. A negative correlation coefficient applies when increases in one source are accompanied by decreases in the other.

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[8] The analysis procedure is based on training materials developed and presented by Integrated Sciences Group (see References).

1.1.9   Covariance. The expected value of the product of the deviations of two random variables from their respective means. The covariance of two *independent* variables is zero.

1.1.10 Coverage Factor. A multiplier used to express an error limit or expanded uncertainty as a multiple of the standard uncertainty.

1.1.11 Cumulative Distribution Function. A mathematical function whose values $F(x)$ are the probabilities that a random variable assumes a value less than or equal to x. Synonymous with Distribution Function.

1.1.12 Degrees of Freedom. A statistical quantity that is related to the amount of information available about an uncertainty estimate. The degrees of freedom signifies how "good" the estimate is and serves as a useful statistic in determining appropriate coverage factors and computing confidence limits and other decision variables.

1.1.13 Direct Measurements. Measurements in which a measuring parameter X directly measures the value of a subject parameter Y (i.e., X measures Y). In direct measurements, the value of the subject parameter is obtained directly by measurement and is not determined by computing its value from the measurement of other variables or quantities.

1.1.14 Distribution Variance. The mean square dispersion of a distribution about its mean or mode value. See also *Variance*.

1.1.15 Effective Degrees of Freedom. The degrees of freedom for combined uncertainties computed from the Welch-Satterthwaite formula.

1.1.16 Error Equation. An algebraic expression that defines the total error in the value of a subject parameter in terms of all relevant component errors.

1.1.17 Error Limits. Bounding values that are expected to contain the error from a given source with some specified level of probability or confidence.

1.1.18 Error Source. A parameter, variable or constant that can contribute error to the determination of the value of a subject parameter. Examples include: measuring parameter bias, random error, resolution error, operator bias, computation error and environmental factors error.

1.1.19 Error Source Correlation. See Correlation Analysis

1.1.20 Error Source Uncertainty. The uncertainty in the error of a given source.

1.1.21 Expanded Uncertainty. A multiple of the standard uncertainty reflecting either a specified confidence level or arbitrary coverage factor.

1.1.22 Heuristic Estimate. An estimate resulting from accumulated experience and/or technical knowledge concerning the uncertainty of an error source.

1.1.23 Independent Error Sources. Error sources that are statistically independent. Two error sources are statistically independent if one does not exert an influence over the other or if both are not consistently influenced by a common agency. See also *Statistical Independence*.

1.1.24 Level of Confidence. See Confidence Level.

1.1.25 Mean Square Error. See Variance.

1.1.26 Mean Value. *Sample Mean*: The average value of a measurement sample. *Population Mean*: The expectation value for measurements sampled from a population.

1.1.27 <u>Measurand</u>. According to Annex B, Section B.2.9 of the ISO GUM, the measurand is defined as "the particular quantity subject to measurement."

1.1.28 <u>Measurement Error</u>. The difference between the measured value of a parameter and its true value.

1.1.29 <u>Measurement Process Errors</u>. Measurement process errors refer to errors resulting from the measurement process (e.g., measuring parameter bias, operator bias, environmental factors, …). Measurement process errors are the basic elements of uncertainty analysis. Once these fundamental error sources have been identified, then we can begin to develop uncertainty estimates.

1.1.30 <u>Measurement Process Uncertainties</u>. See Error Source Uncertainty.

1.1.31 <u>Measurement Uncertainty</u>. The uncertainty in a measurement or in the error in the measurement.

1.1.32 <u>Measuring Parameter</u>. Attribute of a measuring device that is used to obtain information that quantifies the value of the subject parameter.

1.1.33 <u>Median Value</u>. (1) The value that divides an ordered sample of data in two equal portions. (2) The value for which the distribution function of a random variable is equal to one-half. (3) A point of discontinuity such that the distribution function immediately below the point is less than one-half and the distribution function immediately above the point is greater than one-half.

1.1.34 <u>Mode Value</u>. The value of a parameter most often encountered or measured. Sometimes synonymous with the nominal value or design value of a parameter.

1.1.35 <u>Multivariate Measurements</u>. Measurements in which the subject parameter is a computed quantity based on measurements of two or more attributes or parameters.

1.1.36 <u>Nominal Value</u>. The designated or published value of an artifact or parameter. It may also sometimes refer to the mode value of an artifact or parameter.

1.1.37 <u>Parameter</u>. Often thought of as a specified aspect or feature of an instrument or item. In general, however, a parameter does not have to be a "toleranced" quantity or value. See also Attribute.

1.1.38 <u>Population</u>. The total set of possible values for a random variable under consideration.

1.1.39 <u>Population Mean</u>. It is the expectation value of a random variable described by a population distribution.

1.1.40 <u>Probability</u>. The likelihood of the occurrence of a specific event or value from a population of events or values.

1.1.41 <u>Probability Density Function</u>. A mathematical function that describes the relative frequency of occurrence of the values of a random variable.

1.1.42 <u>Repeatability</u>. The closeness of the agreement between the results of successive measurements of the value of a parameter carried out under the same measurement conditions. Repeatability conditions include: the same measurement procedure, the same observer, the same measuring instrument used under the same conditions, the same location, and repetition over a short period of time.

1.1.43 <u>Sample</u>. A collection of values drawn from a population. Typically, inferences about a population are made from the sample. Therefore, the sample must be statistically representative of the population.

1.1.44 <u>Sample Histogram</u>. A bar chart showing the relative frequency of occurrence of sampled data.

1.1.45 <u>Sample Mean</u>. The arithmetic average of the measurements of a sample.

1.1.46 <u>Sample Size</u>. The number of measured values that comprise a sample.

1.1.47 <u>Standard Deviation</u>. The square root of the variance of a sample or population of values. A quantity that represents the spread of values about a mean value. In statistics, the second moment of a distribution.

1.1.48 <u>Standard Uncertainty</u>. A statistic representing spread or uncertainty in the value of a parameter or error source. If determined statistically from sampled data, the standard uncertainty is equal to the sample standard deviation.

1.1.49 <u>Statistical Independence</u>. A property that describes two error sources as being uncorrelated. See also Independent Error Sources.

1.1.50 <u>Subject Parameter</u>. An attribute whose value we seek to obtain from a measurement or set of measurements.

1.1.51 <u>Tolerance Limits</u>. Limits that bound acceptable parameter values.

1.1.52 <u>True Value</u>. The value that would be obtained by a perfect measurement. True values are by nature indeterminate.

1.1.53 <u>Type A Estimates</u>. Uncertainty estimates obtained by the statistical analysis of a sample of data.

1.1.54 <u>Type B Estimates</u>. Uncertainty estimates obtained by heuristic means.

1.1.55 <u>Uncertainty</u>. See Standard Uncertainty.

1.1.56 <u>Uncertainty Component</u>. A contribution to total combined uncertainty from an error source.

1.1.57 <u>Variance</u>. (1) Population: The expectation value for the square of the difference between the value of a variable and the population mean. (2) Sample: A measure of the spread of a sample equal to the sum of the squared observed deviations from the sample mean divided by the degrees of freedom for the sample. Also referred to as the mean square error.

1.2 Defining the Measurement Process

The first step in any uncertainty analysis is to identify the physical quantity whose value is estimated via measurement. This quantity, sometimes referred to as the "measurand," may be a directly measured value, such as the weight of a 1 gm mass or the output of a voltage reference. Alternatively, the quantity may be indirectly determined through the measurement of other variables, as in the case of estimating the volume of a cylinder by measuring its length and diameter. The former type of measurements are called "direct measurements," while the latter are called "multivariate measurements."

For multivariate measurements, it is important to develop an equation that defines the mathematical relationship between the quantity of interest and the measured variables. For the cylinder volume example, this mathematical equation would be expressed as

$$V = \pi \left(\frac{D}{2} \right)^2 L \qquad (1\text{-}1)$$

where

V = cylinder volume in units of interest

D = cylinder diameter in appropriate units

L = cylinder length in appropriate units

π = the ratio of the circumference of a circle to its diameter = 3.14159

At this stage of the analysis, it is also useful to briefly describe the test setup, environmental conditions, technical information about the instruments, reference standards, or other equipment used and the procedure for obtaining the measurement(s). This information will help identify the measurement process errors.

1.3 Errors and Distributions

In any given measurement scenario, each measured quantity is a potential source of error. For example, errors in the length and diameter measurements will contribute to the overall error in the estimation of the cylinder volume. Therefore, the cylinder volume equation can be expressed as

$$V_0 + \varepsilon_V = \pi \left(\frac{D_0 + \varepsilon_D}{2} \right)^2 (L_0 + \varepsilon_L) \qquad (1\text{-}2)$$

where

V_0 = true or nominal cylinder volume

D_0 = true or nominal cylinder diameter

L_0 = true or nominal cylinder length

ε_V = error in the cylinder volume measurement

ε_D = error in the cylinder diameter measurement

ε_L = error in the cylinder length measurement

By rearranging equation (1-2), we obtain an algebraic expression for the cylinder volume error.

$$\varepsilon_V = \pi \left(\frac{D_0 + \varepsilon_D}{2} \right)^2 (L_0 + \varepsilon_L) - V_0$$

$$= \pi \left(\frac{D_0 + \varepsilon_D}{2} \right)^2 (L_0 + \varepsilon_L) - \pi \left(\frac{D_0}{2} \right)^2 L_0$$

$$= \frac{\pi}{4} \left(D_0^2 + 2D_0 \varepsilon_D + \varepsilon_D^2 \right) (L_0 + \varepsilon_L) - \pi \left(\frac{D_0}{2} \right)^2 L_0 \tag{1-3}$$

$$= \frac{\pi}{4} \left(L_0 D_0^2 + 2L_0 D_0 \varepsilon_D + L_0 \varepsilon_D^2 + D_0^2 \varepsilon_L + 2D_0 \varepsilon_D \varepsilon_L + \varepsilon_D^2 \varepsilon_L \right) - \pi \left(\frac{D_0}{2} \right)^2 L_0$$

The terms, $L_0 \varepsilon_D^{\,2}$, $2D_0 \varepsilon_D \varepsilon_L$ and $\varepsilon_D^{\,2} \varepsilon_L$, are referred to as second order terms and are considered to be small compared to the other first order terms in equation (1-3). Neglecting these terms, we can express the cylinder volume error equation in a simpler form.

$$\varepsilon_V = \frac{\pi}{4} \left(L_0 D_0^2 + 2L_0 D_0 \varepsilon_D + D_0^2 \varepsilon_L \right) - \pi \left(\frac{D_0}{2} \right)^2 L_0 \tag{1-4}$$

Rearranging equation (1-4), we can further simplify the equation for ε_V.

$$\varepsilon_V = \pi \left(\frac{D_0}{2} \right)^2 \varepsilon_L + \pi L_0 \frac{D_0}{2} \varepsilon_D + \pi \left(\frac{D_0}{2} \right)^2 L_0 - \pi \left(\frac{D_0}{2} \right)^2 L_0$$

$$= \pi \left(\frac{D_0}{2} \right)^2 \varepsilon_L + \pi L_0 \frac{D_0}{2} \varepsilon_D \tag{1-5}$$

The coefficients for ε_L and ε_D in equation (1-5) are actually the partial derivatives of V with respect to L and D.

$$\frac{\partial V}{\partial L} = \pi \left(\frac{D}{2} \right)^2 \quad \text{and} \quad \frac{\partial V}{\partial D} = \pi L \left(\frac{D}{2} \right)$$

Therefore, the cylinder volume error can be expressed as

$$\varepsilon_V = \left(\frac{\partial V}{\partial L} \right) \varepsilon_L + \left(\frac{\partial V}{\partial D} \right) \varepsilon_D \tag{1-6}$$

where the partial derivatives are sensitivity coefficients that determine the relative contribution of the errors in length and diameter to the total error.

The errors in length and diameter are the sum of the errors encountered during the measurement process and can be expressed as

$$\varepsilon_L = \varepsilon_{L1} + \varepsilon_{L2} + ... + \varepsilon_{Ln}$$

$$\varepsilon_D = \varepsilon_{D1} + \varepsilon_{D2} + ... + \varepsilon_{Dn}$$

where the numbered subscripts signify the different measurement process errors. The errors most often encountered in making measurements include, but are not limited to the following:

- Measurement Bias
- Random or Repeatability Error
- Resolution Error
- Digital Sampling Error
- Computation Error
- Operator Bias
- Environmental Factors Error
- Stress Response Error

Measurement process errors are the basic elements of uncertainty analysis. Once these fundamental error sources have been identified, we can begin to develop uncertainty estimates.

Another important aspect of the uncertainty analysis process is the fact that measurement errors can be characterized by statistical distributions. The statistical distribution for a type of measurement error is a mathematical description that relates the frequency of occurrence of values with the values themselves. Error distributions include, but are not limited to normal, lognormal, uniform (rectangular), triangular, quadratic, cosine, exponential, u-shaped, trapezoidal, and student's t.

Each distribution is characterized by a set of statistics. The statistics most often used in uncertainty analysis are the mean, or the mode, and the standard deviation. With the lognormal distribution, a limiting value and the median value are also used.

UncertaintyAnalyzer automatically computes the distribution statistics in response to data entered by the user. Alternatively, the user can select the desired error distribution from a drop-down list. A brief description of these distributions is given in the following subsections.

1.3.1 <u>Normal Distribution</u>. When obtaining a Type A uncertainty estimate, we compute a standard deviation from a sample of values. For example, we estimate uncertainty due to random error by computing the standard deviation for a sample of repeated measurements of a given value. We also obtain a sample size. The sample standard deviation is an estimate of the standard deviation for the population from which the sample was drawn. Except in rare cases, we assume that this population follows the normal distribution.

Why do we assume a normal distribution? The primary reason is because this is the distribution that either represents or approximates what we frequently see in the physical universe. It can be derived from the laws of physics for such phenomena as the diffusion of gases and is applicable to instrument parameters subject to random stresses of usage and handling. It is also often applicable to equipment parameters emerging from manufacturing processes.
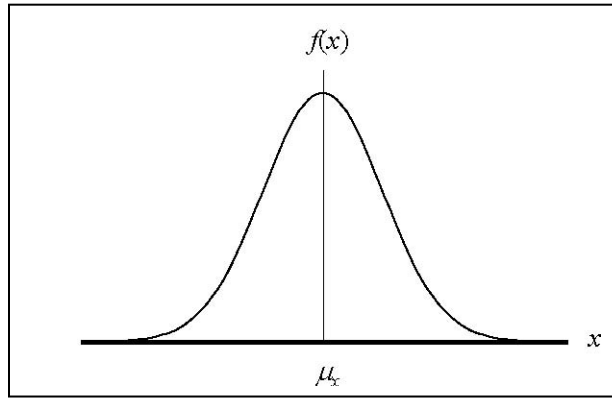
Figure 1-1. The Normal Distribution.

The probability density function for the normal distribution is given in equation (1-7). The variable μ is the population mean and the variable σ is the population standard deviation.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad (1-7)$$

In applying the normal distribution, an uncertainty estimate is obtained from containment limits and a containment probability. The use of the normal distribution is appropriate in cases where the above considerations apply and the limits and probability are at least approximately known.

The extent to which this knowledge is approximate determines the degrees of freedom of the uncertainty estimate. The degrees of freedom and the uncertainty estimate can be used in conjunction with the Student's t distribution to compute confidence limits.

Let $\pm a$ represent the known containment limits and let p represent the containment probability. Then an estimate of the standard deviation of the population of errors or deviations is obtained from equation (1-8).

$$u = \frac{a}{\Phi^{-1}\left(\dfrac{1+p}{2}\right)} \qquad (1-8)$$

The inverse normal distribution function, $\Phi^{-1}()$, can be found in statistics texts and in most spreadsheet programs.

If only a single containment limit is applicable, such as with single-sided tolerances, the appropriate expression is given in equation (1-9).

$$u = \frac{a}{\Phi^{-1}(p)} \qquad (1-9)$$

1.3.2   <u>Lognormal Distribution</u>. The lognormal distribution can often be used to estimate the uncertainty in equipment parameter bias in cases where the tolerance limits are asymmetric. It is also used in cases where a physical limit is present that lies close enough to the nominal or mode

value to skew the parameter bias probability density function in such a way that the normal distribution is not applicable.
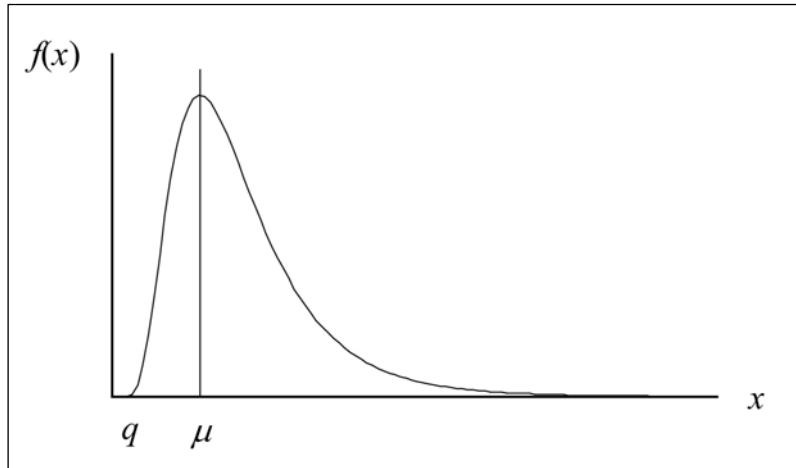
Figure 1-2. The Lognormal Distribution.

The probability density function for the lognormal distribution is given in equation (1-10). The variable q is a physical limit for x, the variable m is the population median and the variable μ is the population mode. The variable σ is not the population standard deviation. Rather, σ is referred to as the "shape parameter." The accompanying graphic shows a case where $\mu = 10$, $q = 9.6207$, $\sigma = 0.52046$, and $m = 13.1117$. The computed standard deviation for this example is $u = 0.3176$.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}\,|x-q|} \exp\left\{ -\left[\ln\left(\frac{x-q}{(\mu-q)m}\right)\right]^2 \Big/ 2\sigma^2 \right\} \tag{1-10}$$

Uncertainty estimates (standard deviations) for the lognormal distribution are obtained by numerical iteration.

1.3.3   <u>Exponential Distribution</u>. We sometimes encounter cases where there exists a definable upper or lower bound to the values (or errors) attainable to a parameter with a single-sided upper or lower tolerance limit. In most instances, the distribution to apply to these cases is the lognormal distribution. This distribution is characterized by a nominal or "mode" value and, as indicated, a bounding physical limit.

It is possible that, for some parameters, the bounding limit and the mode value are equal. If so, then the lognormal distribution suffers from a mathematical discontinuity that makes it inappropriate as the distribution of choice. To handle such parameters, we employ the exponential distribution. A plot of this distribution is shown below for a parameter whose mode value μ is less than its tolerance limit L.

Figure 1-3. The Exponential Distribution.

The probability density function for the exponential distribution is given in equation (1-11).

$$f(x) = \begin{cases} \lambda \, e^{-\lambda \, |x - \mu|}, & |x - \mu| \geq 0 \\ 0, & otherwise \end{cases} \qquad (1\text{-}11)$$

The absolute value for x – μ is used to accommodate cases where the mode is greater than the tolerance limit, as depicted in Figure 1-4.
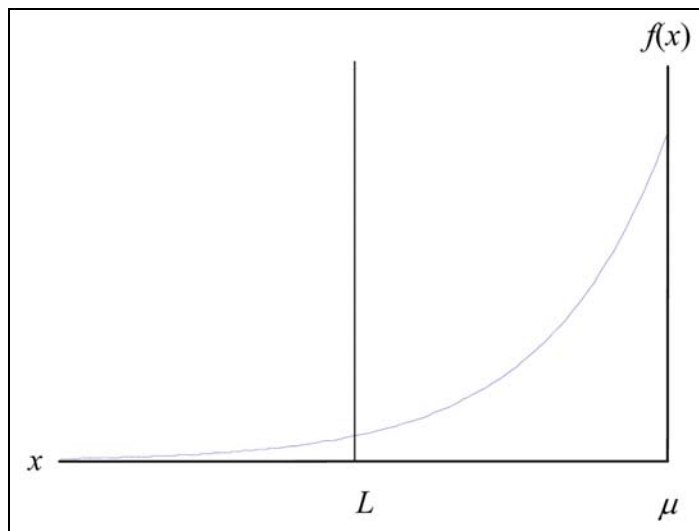
Figure 1-4. Left-Handed Exponential Distribution.

The standard deviation or standard uncertainty estimate for the exponential distribution is obtained from the square root of x. Employing the probability density function for the distribution, it can be shown that this uncertainty is given by equation (1-12).

$$u = \frac{1}{\lambda} \tag{1-12}$$

1.3.4   Quadratic Distribution. A distribution that eliminates the abrupt change at the zero point, does not exhibit unrealistic linear behavior and satisfies the need for a central tendency is the quadratic distribution.
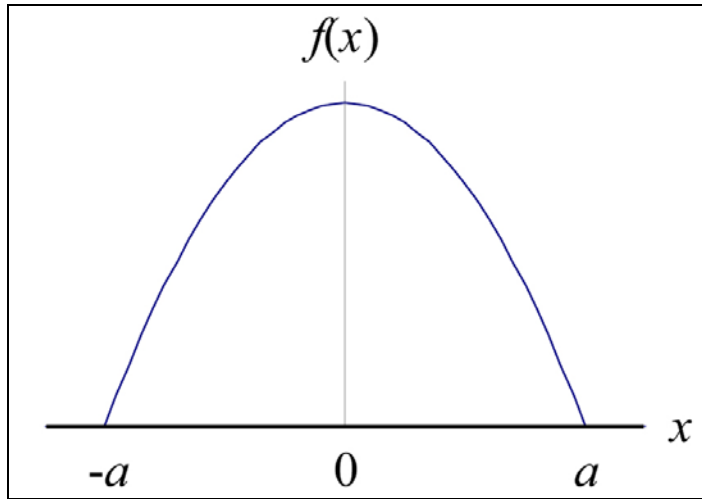
Figure 1-5. The Quadratic Distribution.

This distribution is defined by the probability density function given in equation (1-13).

$$f(x) = \begin{cases} \frac{3}{4a}\left[1 - (x/a)^2\right], & -a \leq x \leq a \\ 0 & , \ otherwise \end{cases} \tag{1-13}$$

Obtaining the minimum bounding limits $\pm a$ when a containment probability p and containment limits $\pm L$ are known involves solving the cubic equation (1-14).

$$L^3 - 3aL + 2a^3 p = 0, \quad L \leq a \tag{1-14}$$

The solution can be obtained numerically. UncertaintyAnalyzer contains a routine that solves for a, along with other parameters that represent more robust incarnations of the quadratic distribution. The iterative algorithm is given in equations (1-15) through (1-17).

$$a_i = a_{i-1} - F / F^{'} \tag{1-15}$$

where

$$F = L^3 - 3aL + 2a^3 p \tag{1-16}$$

and

$$F' = -3L + 6a^2 p \qquad (1-17)$$

and a_i is the value obtained at the ith iteration.

The standard deviation, or standard uncertainty, estimate for the quadratic distribution is determined from equation (1-18).

$$u = \frac{a}{\sqrt{5}} \qquad (1-18)$$

1.3.5 <u>Cosine Distribution</u>. While the quadratic distribution eliminates discontinuities within the bounding limits, it rises abruptly at the limits. And, even though the quadratic distribution has wider applicability than either the triangular or uniform distribution, this feature nevertheless diminishes its physical validity. The cosine distribution overcomes this shortcoming, exhibits a central tendency, and can be determined from minimum containment limits.
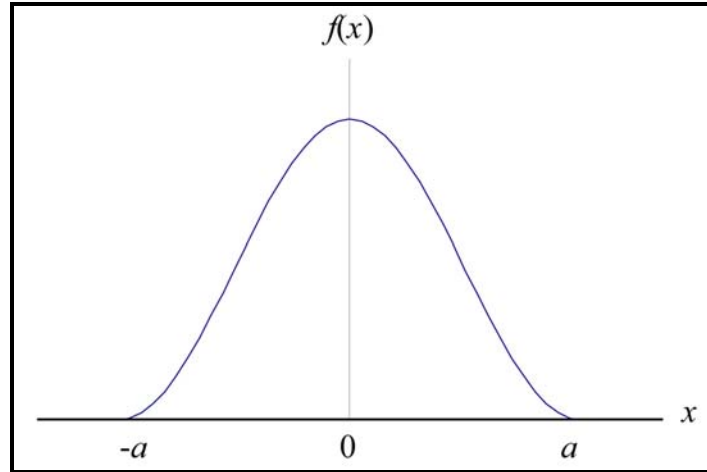
Figure 1-6. The Cosine Distribution.

The probability density function for the cosine distribution is given in equation (1-19).

$$f(x) = \begin{cases} \dfrac{1}{2a}\left[1 + \cos\left(\dfrac{\pi x}{a}\right)\right], & -a \le x \le a \\ 0, & , \ otherwise \end{cases} \qquad (1-19)$$

Solving for a when a containment probability and containment limits $\pm L$ are given quires applying numerical iterative method to the expression given in equation (1-20).

$$\frac{a}{\pi}\sin(\pi L / a) - ap + L = 0, \quad L \le a \qquad (1-20)$$

The solution algorithm has been incorporated into UncertaintyAnalyzer. The iterative algorithm is given in equations (1-21) through (1-23).

$$a_i = a_{i-1} - F / F'$$
(1-21)

where

$$F = \frac{a}{\pi} \sin(\pi L / a) - ap + L$$
(1-22)

and

$$F' = \frac{1}{\pi} \sin(\pi L / a) - \frac{L}{a} \cos(\pi L / a) - p$$
(1-23)

and a_i is the value obtained at the ith iteration.

The standard deviation or standard uncertainty estimate for the cosine distribution is obtained from the expression given in equation (1-24).

$$u = \frac{a}{\sqrt{3}} \sqrt{1 - \frac{6}{\pi^2}}$$
(1-24)

The value of u for the cosine distribution translates to roughly 63% of the value obtained using the uniform distribution.

1.3.6 <u>U-Shaped Distribution</u>. The U distribution applies to sinusoidal RF signals incident on a load. Another application for this distribution would be environmental temperature control in a laboratory or test chamber.

The probability density function for the U-shaped distribution is given in equation (1-25).

$$f(x) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2 - x^2}}, & -a < x < a \\ 0, & otherwise \end{cases}$$
(1-25)

The parameter a represents the maximum signal amplitude.
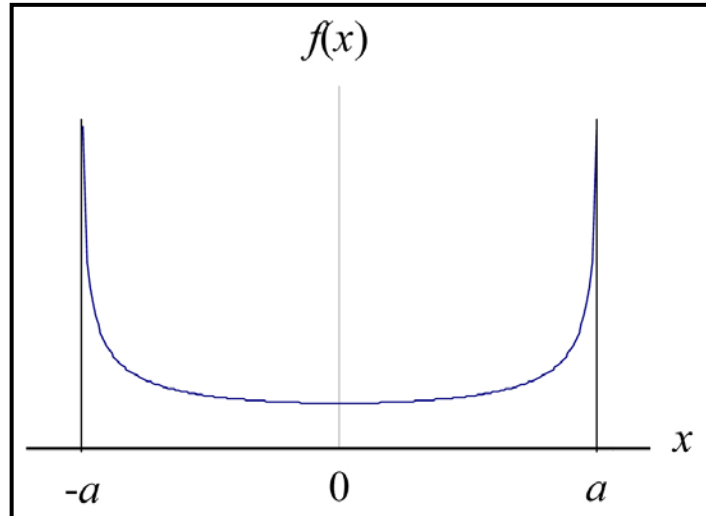
Figure 1-7. The U-Shaped Distribution.

If containment limits $\pm L$ and a containment probability p are known, the parameter a can be computed from equation (1-26).

$$a = \frac{L}{\sin(\pi\, p\, /\, 2)}, \quad L \leq a \qquad (1\text{-}26)$$

The standard deviation or standard uncertainty for the U-shaped distribution is estimated from equation (1-27).

$$u = \frac{a}{\sqrt{2}} \qquad (1\text{-}27)$$

The value of u for the U-shaped distribution translates to roughly 122% of the value obtained using the uniform distribution.

1.3.7   <u>Uniform (Rectangular) Distribution</u>. This uniform distribution has 100% containment limits and the probability of obtaining a value within these limits is equally probable.
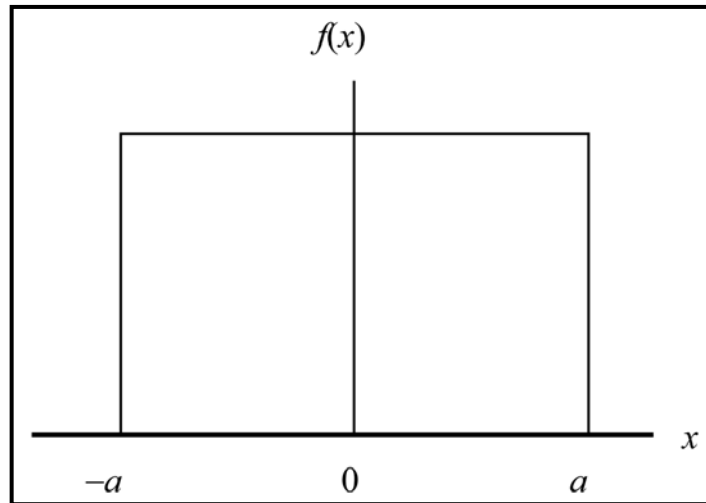
Figure 1-8. The Uniform Distribution.

The probability density function for the uniform distribution is given in equation (1-28).

$$f(x) = \begin{cases} \dfrac{1}{2a}, & -a \leq x \leq a \\ 0, & otherwise \end{cases} \qquad (1\text{-}28)$$

The containment limits of the distribution are $\pm a$. The probability of lying between $-a$ and a is constant. The probability of lying outside $\pm a$ is zero.

The standard deviation, or standard uncertainty, for the uniform distribution is obtained from equation (1-29).

$$u = \frac{a}{\sqrt{3}} \qquad (1\text{-}29)$$

1.3.8 <u>Triangular Distribution</u>. The triangular distribution has been proposed for use in cases where the containment probability is 100%, but there is a central tendency for values of the variable of interest. The triangular distribution is the simplest distribution possible with these characteristics. The triangular distribution sometimes applies to parameter values immediately following test or calibration.
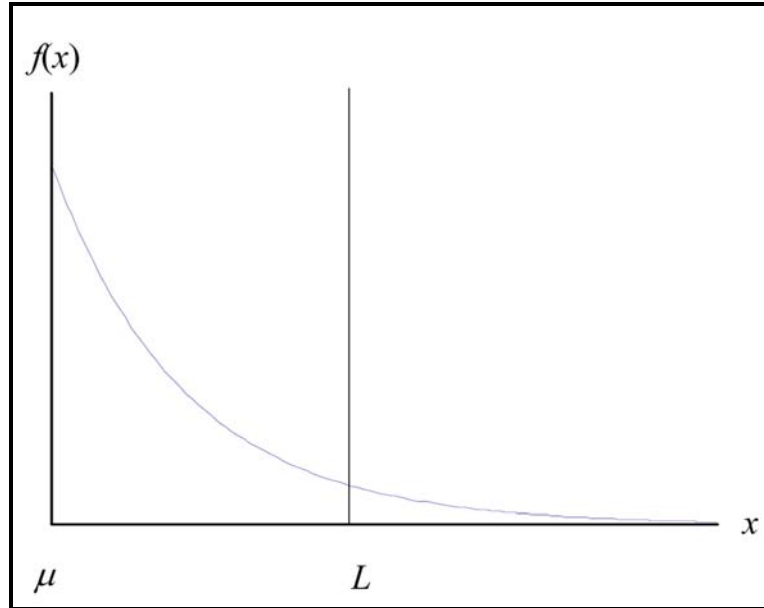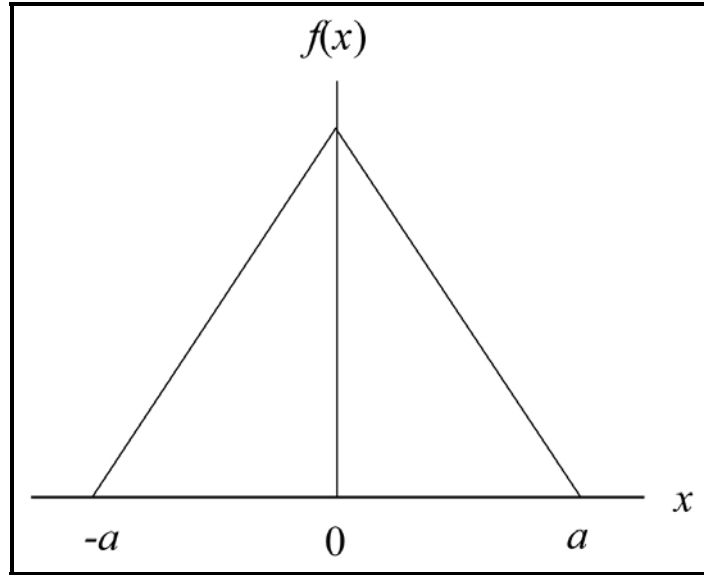
Figure 1-9. The Triangular Distribution.

The probability density function for the triangular distribution is given in equation (1-30).

$$f(x) = \begin{cases} (x+a)/a^2, & -a \leq x \leq 0 \\ (a-x)/a^2, & 0 \leq x \leq a \\ 0, & otherwise \end{cases} \qquad (1\text{-}30)$$

In cases where a containment probability $p < 1$ can be determined for limits $\pm L$, where $L < a$, the limits of the distribution are given by equation (1-31).

$$a = \frac{L}{1 - \sqrt{1-p}}, \quad L \leq a \qquad (1\text{-}31)$$

Apart from representing post-test distributions under certain restricted conditions, the triangular distribution has limited applicability to physical errors or deviations. While it does not suffer from the uniform probability criterion, as does the uniform distribution, it nevertheless displays abrupt transitions at the bounding limits and at the zero point, which are physically unrealistic in most instances. In addition, the linear increase and decrease in behavior is somewhat fanciful for a probability density function.

The standard deviation, or standard uncertainty, for the triangular distribution is obtained from equation (1-32).

$$u = \frac{a}{\sqrt{6}} \qquad (1\text{-}32)$$

Like the uniform distribution, using the triangular distribution requires the establishment of minimum containment limits $\pm a$. The same reservations apply in this regard to the triangular distribution as to the uniform distribution.

1.3.9 <u>Student's t Distribution</u>. If the underlying distribution is normal, and a Type A estimate and degrees of freedom are available, confidence limits for measurement errors or parameter deviations may be obtained using the Student's t distribution. This distribution is available in statistics textbooks and popular spreadsheet applications.
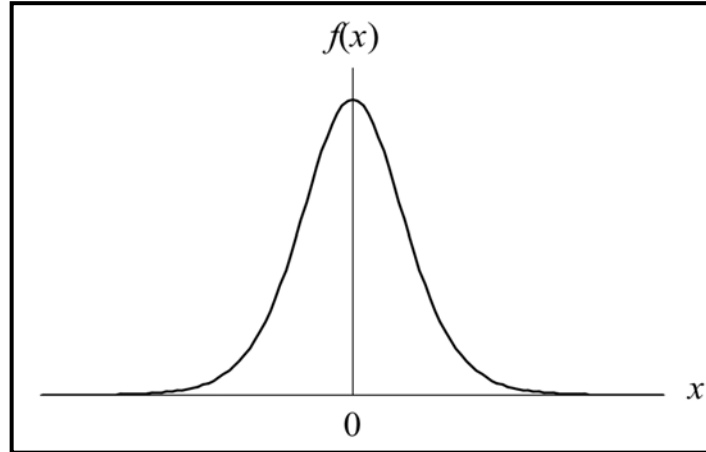
Figure 1-10. Student's t Distribution.

The probability density function for the student's t distribution is given in equation (1-33). The variable v is the degrees of freedom and the parameter $\Gamma()$ is the gamma function. The degrees of freedom quantifies the amount of knowledge used in estimating uncertainty.

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi v}\,\Gamma\left(\frac{v}{2}\right)}(1+x^2/v)^{-(v+1)/2} \qquad (1\text{-}33)$$

For Type A estimates the degrees of freedom is simply the sample size, n, minus one, as shown in equation (1-34).

$$v = \text{n} - 1 \qquad (1\text{-}34)$$

The knowledge used in estimating uncertainty is incomplete if the minimum contain limits ± a for the student's t distribution are approximate and the containment probability p is estimated from recollected experience (i.e., Type B). Therefore, the degrees of freedom associated with a Type B estimate is not infinite.

If the degrees of freedom variable is finite but unknown, the uncertainty estimate cannot be rigorously used to develop confidence limits, perform statistical tests or make decisions. This limitation has often precluded the use of Type B estimates as statistical quantities and has led to the misguided practice of using fixed coverage factors.

Fortunately, the ISO GUM provides an expression for obtaining the approximate degrees of freedom for Type B estimates. However, the expression involves the use of the variance in the uncertainty estimate, and a method for obtaining this variance has been lacking until recently. A rigorous method for obtaining this quantity has been implemented into UncertaintyAnalyzer.

The procedure is to first estimate the uncertainty for the normal distribution and then estimate the degrees of freedom from the expression given in equation (1-35).

$$V_B \cong \frac{1}{2}\left(\frac{\sigma^2(u)}{u^2}\right)^{-1}$$

$$\cong \frac{3\phi^2 a^2}{2\phi^2(\Delta a)^2 + \pi a^2 e^{\phi^2}(\Delta p)^2} \ , \tag{1-35}$$

where

$$\phi = \Phi^{-1}\left(\frac{1+p}{2}\right) \tag{1-36}$$

The variables Δa and Δp represent "give or take" values for the containment limits and containment probability, respectively.

At first glance, equation (1-35) may seem to be anything but rigorous. However, several data input formats have been incorporated into UncertaintyAnalyzer that make the process of estimating Δa and Δp thorough.

Once the degrees of freedom has been obtained, the Type B estimate may then be combined with other estimates, and the degrees of freedom for the combined uncertainty can be determined using the Welch-Satterthwaite formula outlined in Annex G, Section 4.2 of the ISO GUM. If the underlying distribution for the combined estimate is normal, the t distribution can be used to develop confidence limits and perform statistical tests.

For a given confidence or containment limits $\pm L$ and corresponding degrees of freedom, the standard uncertainty can be estimated using equation (1-37).

$$u = \frac{L}{t_{\frac{\alpha}{2}, \nu}} \tag{1-37}$$

The variable $t_{\alpha/2, \nu}$ is the student's t statistic and the variable $\alpha = 1 - p$, where p is the containment probability or confidence level. In UncertaintyAnalyzer, the student's t statistic is computed iteratively for a given set of $\alpha/2$ and ν.

1.3.10 Choosing the Appropriate Distribution. In general, the three error distributions that have been found to be relevant to most real world measurement applications are the normal, lognormal, and Student's t distributions. Other distributions such as the uniform, triangular, quadratic, cosine, exponential, and U-shaped are also possible, although they are rarely applicable.

Some recommendations for selecting the appropriate distribution for a particular error source are given in the following subsection. An additional subsection is included to illustrate the misguided application of the uniform distribution for Type B uncertainty estimates regardless of the error source. More specific criteria for correctly selecting the uniform distribution and example cases that satisfy this criteria are given in the final two subsections.

1.3.10.1 Recommendations for Selecting Distributions. The normal distribution should be applied as the default distribution, unless information to the contrary is available. For Type B

estimates, the data input formats alluded under the discussion of the Student's t distribution should also be employed to estimate the degrees of freedom.

Apply the lognormal distribution if it is suspected that the distribution of the value of interest is skewed.

In using the normal or lognormal distribution, some effort must be made to estimate a containment probability.

If 100% containment has been observed, then the following is recommended:

- Apply the cosine distribution if the value of interest has been subjected to random usage or handling stress, and is assumed to possess a central tendency.

- Apply either the quadratic or half-cosine distribution, as appropriate, if it is suspected that values are more evenly distributed.

- The triangular distribution may be applicable, under certain circumstances, when dealing with parameters following testing or calibration. The triangular distribution may also be applicable for errors due to linear interpolation of tabulated data.

- Apply the U-shaped distribution if the value of interest is the amplitude of a sine wave incident on a plane with random phase.

- Apply the uniform distribution if the value of interest is the resolution uncertainty of a digital readout. This distribution is also applicable to estimating the uncertainty due to quantization error and the uncertainty in RF phase angle.

1.3.10.2 Blind Acceptance of the Uniform Distribution. Applying the uniform distribution to obtaining Type B uncertainty estimates is a practice that has been gaining ground over the past few years. There are two main reasons for this:

1. Applying the uniform distribution makes it easy to obtain an uncertainty estimate. If the limits $\pm a$ of the distribution are known, the uncertainty estimate, u, is simply computed from dividing the containment value a by the square root of 3.

It should be said that the "ease of use" advantage has been promoted by individuals who are ignorant of methods of obtaining uncertainty estimates for more appropriate distributions and by others who are simply looking for a quick solution. In fairness to the latter group, they sometimes assert that the lack of specificity of information required to use other distributions makes for crude uncertainty estimates anyway, so why not get your crude estimate by intentionally using an inappropriate distribution?

At our present level of analytical development, this argument does not hold water. Since the introduction of the ISO GUM, methods have been developed to systematically and rigorously use distributions that are physically realistic. These methods have been incorporated into UncertaintyAnalyzer.

2. It has been asserted by some that the use of the uniform distribution is recommended in the ISO GUM. This is not true. In fact, most of the methodology of the ISO GUM is based on the assumption that the underlying error distribution is normal. Some of the belief that the uniform distribution is called for in the ISO GUM stems from the fact that several individuals, who have come to be regarded as ISO GUM authorities, have been advocating its use. For clarification on this issue, the reader is referred to Section 4.3 of the ISO GUM.

Another source of confusion is that some of the examples in the ISO GUM apply the uniform distribution in situations that appear to be incompatible with its use. It is reasonable to suppose that much of this is due to the fact that rigorous Type B estimation methods and tools were not available at the time the ISO GUM was published, and the uniform distribution was an "easy out." As stated in item 1 above, the lack of such methods and tools has since been rectified and incorporated into UncertaintyAnalyzer.

The philosophy of indiscriminately using the uniform distribution to compute Type B uncertainty estimates undermines efforts to estimate uncertainties that can be used to perform statistical tests, evaluate measurement decision risks, manage calibration intervals, develop meaningful tolerances and compute viable confidence limits. In other words, apart from providing a number, the uncertainty estimate becomes a useless and potentially expensive commodity.

1.3.10.3 <u>Criteria for Selecting the Uniform Distribution</u>. The use of the uniform distribution is appropriate under a limited set of conditions. These conditions are summarized by three criteria. The first criterion is that we must know a set of minimum bounding limits for the distribution. This is the minimum limits criterion. Second, we must be able to assert that the probability of finding values between these limits is unity. This is the 100% containment criterion. Third, we must be able to demonstrate that the probability of obtaining values between the minimum bounding limits is uniform. This is the uniform probability criterion.

1.3.10.3.1 <u>Minimum Limits Criterion</u>. It is vital that the limits we establish for the uniform distribution are the minimum bounding limits. For instance, if the limits $\pm L$ bound the variable of interest, then so do the limits $\pm 2L$, $\pm 3L$, and so on. Since the uncertainty estimate for the uniform distribution is obtained by dividing the bounding limit by the square root of three, using a value for the limit that is not the minimum bounding value will obviously result in an invalid uncertainty estimate.

This alone makes the application of the uniform distribution questionable in estimating bias uncertainty from such quantities as tolerance limits, for instance. It may be that out-of-tolerances have never been observed for a particular parameter (100% containment), but it is unknown whether the tolerances are minimum bounding limits.

Some years ago, a study was conducted involving a voltage reference that showed that values for one parameter were normally distributed with a standard deviation that was approximately 1/10 of the tolerance limit. With 10-sigma limits, it is unlikely that any out-of-tolerances would be observed. However, if the uniform distribution were used to estimate the bias uncertainty for this item, based on tolerance limits, the uncertainty estimate would be nearly six times larger than would be appropriate. Some might claim that this is acceptable, since the estimate can be considered a conservative one. That may be. However, it is also a useless estimate. This point will be elaborated later.

A second difficulty we face when attempting to apply minimum bounding limits is that such limits can rarely be established on physical grounds. This is especially true when using parameter tolerance limits. It is virtually impossible to imagine a situation where design engineers have somehow been able to precisely identify the minimum limits that bound values that are physically attainable. If we add to this the fact that tolerance limits are often influenced by marketing rather than engineering considerations, equating tolerance limits with minimum bounding limits becomes a very unfruitful and misleading practice.

1.3.10.3.2 <u>100% Containment Criterion</u>. By definition, the establishment of minimum bounding limits implies the establishment of 100% containment. It should be said however, that an uncertainty estimate may still be obtained for the uniform distribution if a containment probability less than 100% is applied. For instance, suppose the containment limits are given as $\pm L$ and the containment probability is stated as being equal to some value p between zero and one. Then, if the uniform probability criterion is met, the limits of the distribution are given by

$$a = \frac{L}{p}, \ L \leq a$$

If the uniform probability criterion is not met, however, the uniform distribution would not be applicable, and we should turn to other distributions.

1.3.10.3.3 <u>Uniform Probability Criterion</u>. As discussed above, establishing minimum containment limits can be a challenging prospect. Harder still is finding real-world measurement error distributions that demonstrate a uniform probability of occurrence between two limits and zero probability of occurrence outside these limits. Except in very limited instances, such as ones discussed in the next section, assuming a uniform probability is just not physically realistic. This is true even in some cases where the distribution would appear to be applicable.

For example, a conjecture has recently been advanced that the distribution of parameters immediately following test or calibration can be said to be uniform. While this seems reasonable at face value, it turns out not to be the case. Because of false accept risk (consumer's risk), such distributions range from approximately triangular to having a "humped" appearance with rolled-off shoulders.

As to whether we can treat parameter tolerance limits as bounds that contain values with probability, we must imagine that, not only has the instrument manufacturer managed to miraculously ascertain minimum bounding limits, but has also juggled physics to such an extent as to make the parameter value's probability distribution uniform between these limits and zero outside them. This would be a truly amazing feat of engineering for most toleranced quantities, especially considering the marketing influence mentioned earlier.

1.3.10.3.4 <u>Cases that Satisfy the Criteria</u>.

1.3.10.3.4.1 <u>Digital Resolution Uncertainty</u>. We sometimes need to estimate the uncertainty due to the resolution of a digital readout. For instance, a three-digit readout might indicate 12.015 V. If the device employs the standard round-off practice, we know that the displayed number is derived from a sensed value that lies between 12.0145 V and 12.0155 V. We also can assert to a very high degree of validity that the value has equal probability of lying anywhere between these two numbers. In this case, the use of the uniform distribution is appropriate, and the resolution uncertainty is

$$u_V = \frac{0.0005 \ V}{\sqrt{3}} = 0.00029 \ V \ .$$

1.3.10.3.4.1 <u>RF Phase Angle</u>. RF power incident on a load may be delivered to the load with a phase angle θ between $-\pi$ and π. In addition, unless there is a compelling reason to believe otherwise, the probability of occurrence between these limits is uniform. Accordingly, the use of the uniform distribution is appropriate. This yields a phase angle uncertainty estimate of

$$u_\theta = \frac{\pi}{\sqrt{3}} \cong 1.814 \,.$$

It is interesting to note that, given the above, if we assume that the amplitude of the signal is sinusoidal, the distribution for incident voltage is the U-shaped distribution.

1.3.10.3.4.1 <u>Quantization Error</u>. When an analog signal is digitized, the sampled signal points are quantized in multiples of a discrete step size. The potential drop (or lack of a potential drop) sensed across each element of an analog to digital (A/D) Converter sensing network produces either a "1" or "0" to the converter. This response constitutes a "bit" in the binary code that represents the sampled value. For ladder-type networks, the position of the bit in the code is determined by the location of its originating network element.

Even if no errors were present in sampling and sensing the input signal, errors would still be introduced by the discrete nature of the encoding process. Suppose, for example, that the full scale signal level (dynamic range) of the A/D Converter is a volts. If n bits are used in the encoding process, then a voltage V can be resolved into 2 discrete steps, each of size $a/2$. The error in the voltage V is thus

$$\varepsilon(V) = V - m\frac{a}{2^n} \,,$$

where m is some integer determined by the sensing function of the D/A Converter.

The containment limit associated with each step is one-half the value of the magnitude of the step. Consequently, the containment limit inherent in quantizing a voltage V is $(1/2)(a/2^n)$, or $a/2^{n+1}$. This is embodied in the expression

$$V_{quantized} = V_{sensed} \pm \frac{a}{2^{n+1}} \,.$$

The uncertainty due to quantization error is obtained from the containment limits and from the assumption that the sensed analog value has equal probability of occurrence between these limits:

$$u_V = \frac{a/2^{n+1}}{\sqrt{3}}.$$
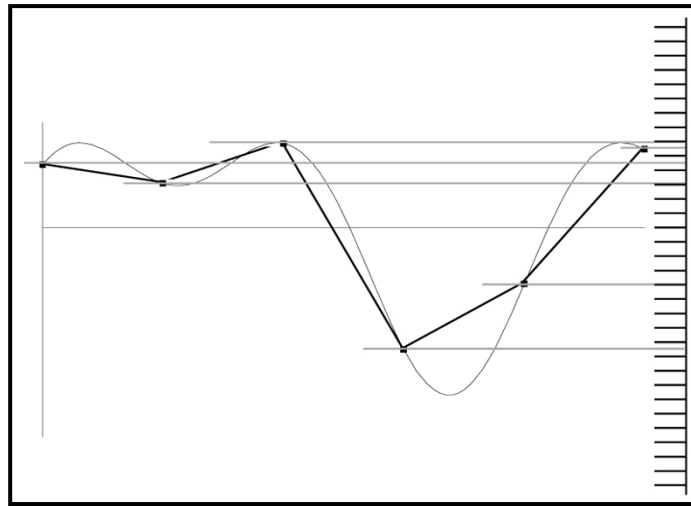
Figure 1-11. Signal Quantization.

1.4 Error and Uncertainty

As previously stated, all measurements are accompanied by error. Our lack of knowledge about the sign and magnitude of measurement error is called measurement uncertainty. To better understand the relationship between measurement error and measurement uncertainty, we will discuss three important axioms that form the basis upon which uncertainties can be estimated. We will also review the variance addition rule, which provides a method for correctly combining uncertainties from different error sources.

Axiom 1 - The uncertainty in a measured value is equal to the uncertainty in the measurement error.

This statement can be shown to be true from the following steps:

1. By definition, measurement error is the difference between the measured value and the true value. Conversely, the measured value is equal to the true value plus the measurement error.

Measured Value = True Value + Measurement Error

2. We define the function for uncertainty in the value x as

Uncertainty (x) = Uncertainty in x

3. The uncertainty in the measured value can then be expressed as

Uncertainty (Measured Value) = Uncertainty (True Value)
+ Uncertainty (Measurement Error)

But the uncertainty in the true value is zero, so

Uncertainty (Measured Value) = Uncertainty (Measurement Error)

Axiom 2 - Measurement errors follow statistical distributions.

This statement basically indicates that errors can be described in such a way that their sign and magnitude have some definable probability of occurrence. With a basic understanding of error distributions and their statistics, we can estimate uncertainties. We begin with the statistical quantity called the variance, which is defined as the mean square dispersion of the distribution about its mean or mode value.

$$\text{var}(x) = \text{Mean Square Dispersion in } x$$

If a variable x follows a probability distribution, described by a probability density function $f(x)$, then the mean square dispersion or variance of the distribution is given by

$$\text{var}(x) = \int_{-\infty}^{\infty} (x - \mu_x)^2 f(x) dx \tag{1-38}$$

where μ_x is the mean of x. Because of the form of this definition, the variance is also referred to as the mean square error.

If a quantity x is a random variable representing a population of measurements, then the variance in x is just the variance in the error in x, which is expressed by the symbol ε_x.

$$\text{var}(x) = \text{var}(x_{true} + \varepsilon_x) = \text{var}(\varepsilon_x) \tag{1-39}$$

Axiom 3 - The uncertainty, u, in a measurement is the square root of the variance in the measurement error.

If x is a measured value, then we can write

$$u_x = \sqrt{\text{var}(x)} = \sqrt{\text{var}(\varepsilon_x)} \tag{1-40}$$

Axiom 3 provides the crucial link between measurement error and measurement uncertainty.

1.5 Quantifying Uncertainty

There are two approaches to estimating variance and uncertainty. Type A estimates involve data sampling and analysis. Type B estimates use engineering knowledge or recollected experience of measurement processes. The basic methods used to estimate Type A and Type B uncertainties are presented herein. Details of how these and other advanced methods are incorporated in the Integrated Systems Group (ISG) Uncertainty Analyzer software are discussed in Chapters 2 and 3.

1.5.1 Type A Estimates. A Type A uncertainty estimate is defined as an estimate obtained from a sample of data. Data sampling involves making repeat measurements of the quantity of interest. It is important that each repeat measurement is independent, representative and taken randomly.

Random sampling is a cornerstone for obtaining relevant statistical information. Thus, Type A estimates usually apply to the uncertainty in repeatability or random error.

Because the data sample is drawn from a population of values, we make inferences about the population from certain sample statistics and from assumptions about the way the population of

values is distributed. A sample histogram can aid in our attempt to picture the population distribution.
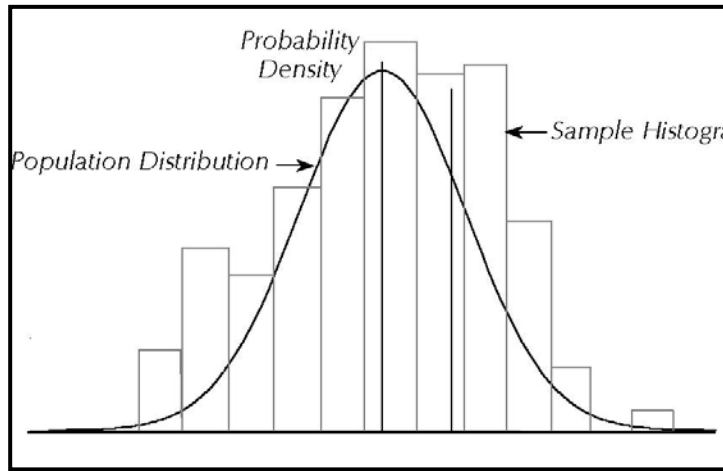
Figure 1-12. Random Error Distribution.

The normal distribution is ordinarily assumed to be the underlying distribution for random errors. When samples are taken, the sample mean and the sample standard deviation are computed and assumed to represent the mean and standard deviation of the population distribution. However, this equivalence is only approximate. To account for this, the Student's t distribution is used in place of the normal distribution in computing confidence limits around sample mean.

The sample mean can be thought of as an estimate of the value that we expect to get when we make a measurement. This "expectation value" is called the population mean, which is expressed by the symbol μ. The sample mean, $\bar{x}$, is obtained by taking the average of the sampled values. The average value is computed by summing the values sampled and dividing them by the sample size, n.

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + ... + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (1\text{-}41)$$

The sample standard deviation provides an estimate of how much the population is spread about the mean value. The sample standard deviation, s_x, is computed by taking the square root of the sum of the squares of sampled deviations from the mean divided by the sample size minus one.

$$s_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad (1\text{-}42)$$

The value n-1 is the degrees of freedom for the estimate, which signifies the number of independent pieces of information that go into computing the estimate. The greater the degrees of freedom, the closer the sample estimate will be to its population counterpart. The degrees of freedom for an uncertainty estimate is useful for establishing confidence limits and other

1-25

decision variables.

We have already stated that the sample standard deviation is an estimate of the uncertainty in a value drawn randomly from its population. However, if the estimate is to represent the uncertainty in the mean value rather than the uncertainty in a single measurement, then the uncertainty in the mean value should be used. The uncertainty in the mean value, $s_{\bar{x}}$, is equal to the standard deviation divided by the square root of the sample size.

$$s_{\bar{x}} = \frac{s_x}{\sqrt{n}} \qquad (1\text{-}43)$$

Once estimates of the sample mean and standard deviation have been obtained, and the degrees of freedom have been noted, it becomes possible to compute limits that bound the sample mean with some specified level of confidence. These limits are called confidence limits and the degree of confidence is called the confidence level.

Confidence limits can be expressed as multiples of the sample standard deviation. For normally distributed samples, this multiple is called the t-statistic. The value of the t-statistic is determined by the desired percent confidence level, C, and the degrees of freedom, v, for the sample standard deviation. Confidence limits around the sample mean are given by

$$\bar{x} \pm t_{\alpha/2,v} \frac{s_x}{\sqrt{n}} \qquad (1\text{-}44)$$

where $\alpha = (1 - C/100)$ and $v = n - 1$.

1.5.2 <u>Type B Estimates</u>. In some cases, we must attempt to quantify the statistics of measurement error distributions by drawing on our recollected experience concerning the values of measured quantities or on our knowledge of the errors in these quantities. Estimates made in this manner are called heuristic or Type B estimates.

Conceivably, a Type B uncertainty estimate could be obtained by just "winging it." The problem is, that most of us do not have a point of reference for abstract quantities such as standard deviation or uncertainties. At best, we have a range of values that we have experienced or are able to surmise. The limiting values that bound these ranges are called containment limits. These limits can be viewed as bounding either measured values or measurement errors.

$$x \text{ lies within } \pm L$$

In working with Type A estimates, we start with sample statistics and work toward developing confidence limits that bound values of a population with a specified confidence level or probability. In making Type B estimates, we apply this process in reverse. We begin with containment limits and a containment probability, estimate the degrees of freedom, and use these quantities to estimate the standard deviation or uncertainty.

$$x \text{ lies within } \pm L \text{ with } C\% \text{ confidence or probability}$$

Containment limits may be estimated from experience or taken from some documented reference, such as manufacturer tolerance limits, stated expanded uncertainties obtained from calibration records or certificates, or statistical process control limits. Containment probability can be obtained from service history data, for example, as the number of observed in-tolerances, $n_{\text{in-tol}}$, divided by the number of calibrations, N.

$$C\% = 100\% \frac{n_{in-tol}}{N}$$

If a heuristic estimate is obtained solely from containment limits and containment probabilities, then the degrees of freedom is usually taken to be infinite. For example, if the measurement error is normally distributed, the uncertainty is computed from the containment limits, $\pm L$, the inverse normal distribution function, $\Phi^{-1}()$, and the containment probability, $p = C/100$. The inverse normal distribution function can be found in statistics texts and in most spreadsheet programs. The appropriate relation between uncertainty and containment limits and containment probability is given in equation (1-45).

$$u = \frac{L}{\Phi^{-1}\left(\frac{1+p}{2}\right)} \tag{1-45}$$

If there is an uncertainty in the containment limits (e.g., $\pm L \pm \Delta L$) or the containment probability (e.g., $\pm p \pm \Delta p$), then it becomes imperative to estimate the degrees of freedom. As with Type A uncertainty estimates, the degrees of freedom quantifies the amount of information that goes into the Type B uncertainty estimate and is useful for establishing confidence limits and other decision variables.

Annex G of the ISO GUM provides a relationship for computing the degrees of freedom for a Type B uncertainty estimate

$$\nu \approx \frac{1}{2}\frac{u^2(x)}{\sigma^2[u(x)]} \approx \frac{1}{2}\left[\frac{\Delta u(x)}{u(x)}\right]^{-2} \tag{1-46}$$

where $\sigma^2[u(x)]$ is the variance in the uncertainty estimate, $u(x)$, and $\Delta u(x)$ is the relative uncertainty in the uncertainty estimate.[9] Hence, the degrees of freedom for a Type B estimate is inversely proportional to the square of the ratio of the uncertainty in the uncertainty divided by the uncertainty.

While this approach is intuitively appealing, the ISO GUM offers no advice about how to determine $\sigma^2[u(x)]$ or $\Delta u(x)$. Fortunately, since the publication of the ISO GUM, a methodology for determining $\sigma^2[u(x)]$ and computing the degrees of freedom for Type B estimates has been developed.

Once the containment limits, containment probability and the degrees of freedom have been established, we can estimate the standard deviation or uncertainty of the distribution of interest. For instance, if the measurement errors are normally distributed, we can construct a t-statistic based on the containment probability and degrees of freedom. The uncertainty estimate is then obtained by dividing the containment limit by the t-statistic, according to equation (1-47).

$$u = \frac{L}{t_{\alpha/2,\nu}} \tag{1-47}$$

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[9] This equation assumes that the underlying error distribution is normal.

1.6 Combining Uncertainties

For purposes of illustration, let us consider a quantity or parameter z that is obtained indirectly from the measurement of the quantities x and y. We will say that z is a linear function of the quantities x and y.

$$z = ax + by$$

where the coefficients a and b are constants.

In this case, we are interested in the uncertainty in z in terms of the uncertainties in the measured quantities x and y. Additionally, measurement errors for x and y are composed of various process errors (e.g., random, bias, resolution, environmental, operator, etc.).

We recall that Axiom 3 states that the uncertainty in the value of an error is equal to the square root of the variance of the error distribution. As a consequence, we can apply the **variance addition rule** to obtain a method for correctly combining uncertainties from different error sources.

1.6.1 <u>Variance Addition Rule</u>. The variance of z can be expressed in terms of the variances of the individual variables, x and y

$$\mathrm{var}(z) = \mathrm{var}(ax + by)$$
$$= a^2\,\mathrm{var}(x) + b^2\,\mathrm{var}(y) + 2ab\,\mathrm{cov}(x, y)$$

where the last term is the **covariance** between x and y.

If we recall Axiom 3 and equation (1-40), we can express the variance of z as shown in equation (1-48).

$$u_z^{\,2} = a^2 u_x^{\,2} + b^2 u_y^{\,2} + 2ab\,\mathrm{cov}(x, y) \qquad (1\text{-}48)$$

If two variables x and y are described by a joint probability density function $f(x,y)$, then the covariance of x and y is given by

$$\mathrm{cov}(x, y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y) f(x, y) dy \qquad (1\text{-}49)$$

where μ_x and μ_y are mean values for x and y, respectively.

The covariance is rarely used explicitly. Instead, we use the **correlation coefficient**, $\rho_{x,y}$, which is defined as

$$\rho_{x,y} = \frac{\mathrm{cov}(x, y)}{u_x u_y} \qquad (1\text{-}50)$$

With this relationship, the variance in the sum of the two quantities x and y is given by equation (1-51).

$$u_z^{\,2} = a^2 u_x^2 + b^2 u_y^2 + 2ab\rho_{x,y} u_x u_y \qquad (1\text{-}51)$$

Equation (1-51) can be generalized to cases where there are n measured quantities $x_1, x_2, \dots x_n$.

$$\text{var}(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i^2 \text{var}(x_i) + 2\sum_{i=1}^{n} \sum_{j>1}^{n} a_i a_j \rho_{ij} u_i u_j$$

$$= \sum_{i=1}^{n} a_i^2 u_i^2 + 2\sum_{i=1}^{n} \sum_{j>1}^{n} a_i a_j \rho_{ij} u_i u_j$$

(1-52)

1.7 Correlating Error Sources

With the variance addition rule, we have a logical approach for combining uncertainties that accounts for correlations between error sources. To assess the impact of the correlated errors on combined uncertainty, let us consider the measurement of a quantity x that involves two error sources ε_1 and ε_2.

$$x = x_{true} + \varepsilon_1 + \varepsilon_2$$

From Axioms 1 and 3 and the variance addition rule, the uncertainty in x is obtained from

$$u_x = \sqrt{\text{var}(x_{true} + \varepsilon_1 + \varepsilon_2)} = \sqrt{\text{var}(\varepsilon_1 + \varepsilon_2)} = \sqrt{u_1^2 + u_2^2 + 2\rho_{1,2} u_1 u_2}$$

The correlation coefficient, $\rho_{1,2}$, for two error sources can range in value from -1 to +1. If the two error sources are statistically independent, then $\rho_{1,2} = 0$ and $u_x = \sqrt{u_1^2 + u_2^2}$. Therefore, uncertainties of statistically independent error sources are combined in a root-sum-square (RSS).

Conversely, if the two error sources are strongly correlated then $\rho_{1,2} = 1$ or -1. If $\rho_{1,2} = 1$, then $u_x = \sqrt{u_1^2 + u_2^2 + 2u_1 u_2} = \sqrt{(u_1 + u_2)^2} = u_1 + u_2$. Therefore, the uncertainties are combined linearly.

When two error sources are strongly correlated and compensate for one another, then $\rho_{1,2} = -1$ and $u_x = \sqrt{u_1^2 + u_2^2 - 2u_1 u_2} = \sqrt{(u_1 - u_2)^2} = |u_1 - u_2|$. Therefore, the combined uncertainty is the absolute value of the difference between the individual uncertainties.

1.7.1 <u>Correlations between Measurement Process Errors</u>. There typically aren't any correlations between measurement process errors for a given quantity. In general it is safe to assume that there are no correlations between the following measurement process errors.
- Random Error and Parameter Bias ($\rho_{ran,bias} = 0$)
- Random Error and Operator Bias ($\rho_{ran,oper} = 0$)
- Parameter Bias and Resolution Error ($\rho_{bias,res} = 0$)
- Parameter Bias and Operator Bias ($\rho_{bias,oper} = 0$)
- Operator Bias and Environmental Factors Error ($\rho_{oper,env} = 0$)
- Resolution Error and Environmental Factors Error ($\rho_{res,env} = 0$)
- Digital Resolution Error and Operator Bias ($\rho_{dres,oper} = 0$)

1.7.2 <u>Accounting for Cross-Correlations</u>. In some instances, the measurement process errors for different quantities may be correlated. For the cylinder volume example, this would occur if the

same device is used to measure the cylinder length and diameter, L and D. In this case, the uncertainty in the measuring parameter bias is the same for both quantities and components.

We account for cross-correlations by developing an expression for the correlation coefficient, $\rho_{L,D}$, between the total uncertainties for each component, u_L and u_D, in terms of the cross-correlations, $\rho_{Li,Dj}$, between the measurement process uncertainties for each component, u_{Li} and u_{Dj}.

$$\rho_{L,D} = \frac{1}{u_L u_D} \sum_{i=1}^{ni} \sum_{j=1}^{nj} \rho_{Li,Dj} u_{Li} u_{Dj} \qquad (1\text{-}53)$$

Accounting for cross-correlations will be discussed further in Chapter 6.

1.8 Degrees of Freedom

When uncertainties are combined, we need to know the degrees of freedom for the total uncertainty. As might be expected, the degrees of freedom for a combined uncertainty estimate is not a simple sum of the degrees of freedom for each uncertainty component. The effective degrees of freedom, v_{eff}, for the total uncertainty, u_T, resulting from the combination of uncertainties u_i and associated degrees of freedom, v_i, for n error sources can be estimated via the Welch-Satterthwaite formula[10] given in Annex G of the ISO GUM.

$$v_{eff} = \frac{u_{T*}^4}{\sum_{i}^{n} \frac{u_i^4}{v_i}} \qquad (1\text{-}54)$$

The combined uncertainty u_{T*} is computed assuming no error source correlations. Consequently, the effective degrees of freedom computed from equation (1-54) is considered a rough estimate.

1.9 Confidence Limits and Expanded Uncertainty

As previously stated, the uncertainty, u, and degrees of freedom, v, can be used to establish confidence limits. These are the upper and lower limits that contain the true value, μ (estimated by the mean value $\bar{x}$), with some specified confidence level or probability, p. Confidence limits are expressed as

$$\bar{x} - t_{\alpha/2,v} u \leq \mu \leq \bar{x} + t_{\alpha/2,v} u \qquad (1\text{-}55)$$

where the multiplier is the t-statistic, $t_{\alpha/2v}$, and $\alpha = 1 - p$.

The ISO GUM defines the term **expanded uncertainty** as "the quantity defining an interval about the result of a measurement that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand." In less obtuse language, the expanded uncertainty is basically defined as a set of limits ($\pm L$) that are expected to contain the true value of the measurand. In this context, the expanded uncertainty, ku, is offered as an approximate confidence limit, in which the coverage factor, k, is used in place of

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[10] This formula is based on the assumption that there are no correlations between error source uncertainties.

the t-statistic.

$$\bar{x} - ku \leq true\, value \leq \bar{x} + ku \qquad (1-56)$$

The introduction of the expanded uncertainty is confusing at best, since it is counterintuitive to think of an uncertainty as having a range. In actual practice, the terms expanded uncertainty and uncertainty are used interchangeably. This, of course, can lead to incorrect inferences and miscommunications. To mitigate this problem, the ISO GUM also introduced the term "standard uncertainty" to help distinguish uncertainty from expanded uncertainty. Unfortunately, confusion over and misapplication of these terms persists.

1.10 Reporting Analysis Results

When reporting the results of an uncertainty analysis, Section 7 of the ISO GUM recommends that the following information be included:

1. The estimated value of the quantity of interest (measurand) and its combined uncertainty and degrees of freedom.

2. The functional relationship between the quantity of interest and the measured components, along with the sensitivity coefficients.

3. The value of each measurement component and its combined uncertainty and degrees of freedom.

4. A list of the measurement process uncertainties and associated degrees of freedom for each component, along with a description of how they were estimated.

5. A list of applicable correlation coefficients, including any cross-correlations between component uncertainties.

It is also a good practice to provide a brief description of the measurement process, including the procedures and instrumentation used, and additional data, tables and plots that help clarify the analysis results.

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CHAPTER 2

TYPE A UNCERTAINTY ESTIMATION

2.0 General

As previously stated, Type A uncertainty estimates are defined as estimates obtained from a sample of data. For example, measurement uncertainty due to random or repeatability error is estimated from a statistical analysis of a sample of measurements.

This chapter discusses how key sample statistics are computed and used to estimate Type A uncertainties.[11] In particular, the methodology focuses on uncertainty estimation for the error due to random variations in repeat measurements made by the subject parameter or measuring parameter during a measurement session.

2.1 Definitions

2.1.1 <u>Adjusted Mean</u>. The value of a parameter or error source obtained by applying a correction factor to a nominal or mean value.

2.1.2 <u>Between Sample Sigma</u>. The standard deviation of the mean values.

2.1.3 <u>Computed Mean Value</u>. The mean value of a parameter determined from a sample of measurements.

2.1.4 <u>Confidence Level</u>. The probability that a set of error limits or containment limits will contain errors for a given error source.

2.1.5 <u>Confidence Limits</u>. Limits that bound errors for a given error source with a specified probability or "confidence."

2.1.6 <u>Coverage Factor</u>. A multiplier used to express an error limit of expanded uncertainty as a multiple of the standard uncertainty.

2.1.7 <u>Degrees of Freedom</u>. A statistical quantity that is related to the amount of information available about an uncertainty estimate and serves as a useful statistic in determining appropriate coverage factors and computing confidence limits and other decision variables. For Type A estimates, the degrees of freedom is the sample size minus one.

2.1.8 <u>Deviation from Nominal</u>. The difference between a parameter's measured value and its nominal value.

2.1.9 <u>Histogram</u>. See Sample Histogram.

2.1.10 <u>Kurtosis</u>. A measure of the "peakedness" of the sample distribution. Normal distributions have a peakedness value of three.

2.1.11 <u>Mean Deviation</u>. The difference between a sample mean value and the nominal value.

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[11] Statistical methods presented herein can be found in various published books including, but not limited to: *Mathematical Statistics and Data Analysis*, Rice. J., Duxbury Press, 1995 and *Probability and Statisitcs*, 2nd Edition, Spiegel, M. R. et. al, McGraw-Hill, 2000.

2.1.12 Mean Value Correction. The correction or adjustment of the computed mean value for an offset due to parameter bias and/or environmental factors.

2.1.13 Measuring Parameter. Attribute of a measuring device that is used to obtain information that quantifies the value of the subject parameter.

2.1.14 Nominal Value. The designated or published value of an artifact or parameter. It may also sometimes refer to the mode value of an artifact or parameter.

2.1.15 Parameter Bias. A systematic deviation of a parameter value from its nominal or indicated value.

2.1.16 Random Error. An error that appears as differences in the measured values of a given artifact or parameter during a measurement session. Sometimes due to random fluctuations in the subject parameter value, the measuring parameter value, and the measurement process.

2.1.17 Repeatability. The closeness of the agreement between the results of successive measurements of the value of a parameter carried out under the same measurement conditions.

2.1.18 Repeatability Error. See random error.

2.1.19 Sample. A collection of values drawn from a population. Typically, inferences about a population are made from the sample. Therefore, the sample must be statistically representative of the population.

2.1.20 Sample Histogram. A bar chart showing the relative frequency of occurrence of sampled data.

2.1.21 Sample Mean. The average value of a measurement sample.

2.1.22 Sample Size. The number of measured values that comprise a sample.

2.1.23 Skewness. A measure of the asymmetry of the sample distribution. A symmetric distribution has zero skewness.

2.1.24 Standard Deviation. The square root of the variance of a sample or population of values. A quantity that represents the spread of values about a mean value. In statistics, the second moment of a distribution.

2.1.25 Standard Uncertainty. A statistic representing spread or uncertainty in the value of a parameter or error source. If determined statistically from sampled data, the standard uncertainty is equal to the sample standard deviation.

2.1.26 Student's t-statistic. Typically expressed as $t_{\alpha,v}$, it denotes the value for which the area under the t-distribution with degrees of freedom, v, is equal to α. A multiplier used to express an error limit or expanded uncertainty as a multiple of the standard uncertainty.

2.1.27 Subject Parameter. An attribute whose value we seek to obtain from a measurement or set of measurements.

2.1.28 t Distribution. A symmetric, continuous distribution characterized by the degrees of freedom parameter. Used to compute confidence limits for normally distributed variables whose standard deviation is based on a finite degrees of freedom. Also referred to as the Student's t distribution.

2.1.29 <u>Uncertainty in the Mean Value</u>. The sample standard deviation divided by the square root of the sample size.

2.1.30 <u>Within Sample Sigma</u>. An indicator of the variation within samples.

2.2 Statistical Sample Analysis

Repeat measurements are entered and analyzed in UncertaintyAnalyzer's Measurement Data Entry Worksheet. The statistical analysis methods employed in this worksheet depend on the type of data entered.

2.2.1 <u>Data Entry Options</u>. There are four data sample types to choose from: Sampled Values, Sampled Cells, Sampled Mean Values, and Mixed. The measurements results data entry table changes depending upon which of these options has been selected.

2.2.1.1 <u>Sampled Values</u>. This data sample option is selected when entering individual repeat measurements. The data can be entered as measured values or deviations from the nominal or specified value of the parameter.

2.2.1.2 <u>Sampled Cells</u>. This data sample option is selected when entering a sampled value that has been observed one or more times. In this case, the value can be entered as a measured value or a deviation from nominal. The corresponding n umber of times that the value was observed is entered under the Number Sampled column.

2.2.1.3 <u>Sampled Mean Values</u>. This data sample option is selected when entering mean values obtained from sets of repeat measurements. The mean values or deviations from the parameter nominal value are entered along with the standard deviation and sample size for each set of repeat measurements.

2.2.1.4 <u>Mixed Samples</u>. This option is selected when a prior sample of data has been used to compute a standard uncertainty that characterizes the repeatability uncertainty of a measurement process. Data are entered on the worksheet in the same way as with the Sampled Values option. The usage of the prior standard uncertainty and the statistics computed from the entered sample are in discussed in sections 2.2.3.3 and 2.2.3.4.

2.2.2 <u>Importing Data</u>

Data can be imported directly into the Measurement Results table from the following external applications:

- Microsoft Access
- dBASE III or IV
- Microsoft Excel 3.0 through 2000
- Lotus 1-2-3 WK1 through WK4
- Hypertext Markup Language (HTML)
- Delimited Text Files
- Open Database Connectivity (ODBC)

Importing is done via UncertaintyAnalyzer's Data Import Profile Screen.

2.2.3 <u>Computed Statistics</u>. The computed statistics vary slightly depending upon which data entry or Sampling Option has been chosen. The computed statistics for all four sampling options include the sample size, sample mean, mean deviation, and standard uncertainty. The user can input a confidence level (%) or coverage factor to compute confidence limits. The display precision (e.g., 1 decimal place, 2 decimal places, ...) for the computed statistics can also be set by the user. Additional statistics for each sampling option are described below.

2.2.3.1 <u>Sampled Values and Sampled Cells Options</u>. The coefficient of skewness and coefficient of kurtosis are also computed for these data entry options. Approximate distributions can be used for the skewness and kurtosis coefficients computed from samples of size n with mean

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad (2\text{-}1)$$

and standard deviation

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} . \qquad (2\text{-}2)$$

The skewness coefficient is given by

$$b_3 \frac{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^3}{s^3} \qquad (2\text{-}3)$$

For $n \geq 10$, the skewness coefficient follows an approximately $N(0, \sigma)$ distribution. A population mean μ and a standard deviation σ can be estimated from a simulated distribution for b_3, obtained by generating m samples of size n. The values of x are obtained as random normal deviates. By experimentation, it has been found that $m = 1000$ produces good results.

With regard to the random number generator, the Rnd function of Visual Basic is adequate, but the Ran2 function described in Numerical Recipes in FORTRAN seems to produce better results.

The coefficient of kurtosis is

$$b_4 \frac{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^4}{s^4} \qquad (2\text{-}4)$$

For $n \geq 10$, the coefficient of kurtosis follows an approximate gamma distribution with the following probability density function:

$$f(x) = (x/b)^{c-1} \frac{e^{-(x/b)}}{b\Gamma(c)} \qquad (2\text{-}5)$$

As with skewness, a population mean μ and a standard deviation σ can be estimated from a simulated distribution for b_4, obtained by generating m samples of size n. The values of x are obtained as random normal deviates. Again, $m = 1000$, used in conjunction with the Ran2 function, produces good results.

The parameters b and c are obtained from

$$c = (\mu / \sigma)^2$$

and

$$b = \mu / c \,.$$

2.2.3.2 <u>Sampled Mean Values Option</u>. When mean values are entered into the Measurement Results table, two additional statistics are computed: 1) Within Sample Sigma and 2) Between Sample Sigma. Before we discuss these quantities, let us first review what the sample mean option entails.

Let us assume that there are k sampled mean values entered into the table. The ith mean value and standard deviation of the ith sample have been computed via a spreadsheet or other program using the following equations

$$\bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij} \tag{2-6}$$

$$s_i = \sqrt{\frac{1}{n_i - 1} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2} \tag{2-7}$$

where

    n_i = sample size for the ith sample

    $\bar{x}_i$ = mean value for the ith sample (i.e., i[th] mean value)

    s_i = standard deviation of ith sample

    x_{ij} = the jth measurement of the ith sample

The Between Sample Sigma, s_b, is the standard deviation of the mean values and is computed as follows

$$s_b = \sqrt{\frac{1}{n_i - 1} \sum_{i=1}^{k} n_i (\bar{x}_i - \bar{x})^2} \tag{2-8}$$

where

    k = number of samples (i.e., number of mean values entered)

    n = total number of measurements (i.e., cumulative of all sample sizes) = $\sqrt{\sum_{i=1}^{k} n_i}$

    $\bar{x}$ = mean value of all measurements (i.e., overall mean value) $= \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_1} x_{ij} = \frac{1}{n} \sum_{i=1}^{k} n_i \bar{x}_i$

The Within Sample Sigma, s_w, is an indicator of the variation within samples and is computed as follows

$$s_w = \sqrt{\frac{1}{n-1} \sum_{i=1}^{k} (n_i - 1) s_i^2} \qquad (2\text{-}9)$$

The standard uncertainty, s, of the sample mean values is given as

$$
\begin{aligned}
s &= \sqrt{\frac{1}{n-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2} \\
&= \sqrt{\frac{1}{n-1} \sum_{i=1}^{k} n_i (\bar{x}_i - \bar{x})^2 + \frac{1}{n-1} \sum_{i=1}^{k} (n_i - 1) s_i^2} \\
&= \sqrt{s_b^2 + s_w^2}
\end{aligned}
\qquad (2\text{-}10)
$$

2.2.3.3 <u>Mixed Sample Option</u>. As previously mentioned, in the Mixed option a prior sample of data has been obtained, along with a standard uncertainty that characterizes a measurement process. A sample of data is entered into the Measurement Results table in the same way the Sampled Values option.

Let the sample size for this sample be designated n and the ith measured value be designated x_i. Then the sample mean and standard deviation (i.e., standard uncertainty) are computed according to

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad (2\text{-}11)$$

and

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} \qquad (2\text{-}12)$$

Let the prior sample size and standard uncertainty be designated n_p and s_p, respectively. The prior standard uncertainty is entered into the Distribution Uncertainty Estimate field and the associated sample size is entered into the Applicable Sample Size field. With the Mixed option, we use these variables to represent the repeatability of the measurement process and to estimated confidence limits for the computed mean value with a confidence level of $(1 - \alpha) \times 100\%$

$$\bar{x} \pm t_{\alpha/2, v} s_p ,$$

where

$$v = n_p - 1$$

and $t_{\alpha/2, v}$ is the Student's t-statistic for a significance level of $\alpha/2$ and v degrees of freedom.

2.2.3.4 <u>Override Sample Size and Uncertainty</u>. In some cases, only one measurement or deviation may be available from a measurement process. If so, then a standard uncertainty will not be computed. However, if a standard uncertainty has been obtained elsewhere that characterizes the repeatability uncertainty in a measurement process, it would be beneficial to be

able to enter it into the Computed Statistic section of the Measurement Data Entry Worksheet, along with the appropriate sample size.

But, if data are entered into this section of the Worksheet, UncertaintyAnalyzer responds by deleting the measurement sample information in the Measurement Results table. To prevent this from happening, the Override Sample Size and Uncertainty box must be checked.

2.2.3.5 Recalc Option. Although UncertaintyAnalyzer automatically computes sample statistics as data are entered the Recalc option was added to provide the user with an alternative means of ensuring that up-to-date computations have been completed.

2.2.3.6 Mean Value Correction. The Measurement Data Entry Worksheet also allows for correcting or adjusting the computed mean value for an offset due to parameter bias and/or environmental factors. The corrected mean value is not displayed on the Measurement Data Entry Worksheet, but is displayed for the Computed Mean Value on the program Main Screen.

If a bias (i.e., systematic offset) from the parameter's nominal value has been input in the Subject Parameter Bias Uncertainty Worksheet, then the user can select whether or not to include the parameter bias correction factor to the computed mean value.

A fixed offset in the subject (or measuring) parameter value or reading can also result from environmental factors (e.g., temperature, humidity, air pressure, ...). The user can select whether or not to adjust the computed mean value by an environmental correction. The environmental correction is estimated using the Measurement Environment Uncertainty Worksheet.

2.2.4 Sample Statistics. The Sample Statistics screen also computes statistics for a user-selected subset of data entry cells. Computed statistics for the sample subset include the mean, mean deviation, standard deviation, skewness and kurtosis. The Sample Statistics screen is accessed via the Tools menu of the Measurement Data Entry Worksheets or the Statistics menu of the Error Source Worksheets.

2.2.5 Normality Testing. UncertaintyAnalyzer's Sample Statistics screen also contains a normality testing feature that can be used to determine if the sampled data can be assumed to be normally distributed. Normality testing in UncertaintyAnalyzer is done using three different tests. A chi-squared "goodness of fit" is performed for samples of size 50 or more. For samples of size 10 or greater, tests are performed to evaluate the skewness and kurtosis of the sample in comparison to what is expected of samples from a normally distributed population.

2.2.6 Outlier Removal. Statistical outliers can be identified and removed from the data samples entered into the Measurement Data Entry Worksheet or Error Source Uncertainty Worksheet by selecting the Outlier Removal option. Outliers are identified using Chauvenet's criterion,[12] which defines acceptable scatter around a mean value $\bar{x}$ for a given sample on n readings and standard deviation s_n.

Chauvenet's criterion specifies that all points should be retained that fall within a band around the mean value that corresponds to a probability of $1 - 1/2n$. The normal distribution is used to determine the number of sample standard deviations that relate to this probability. This "coverage factor" is obtained using the two-tailed inverse normal function $\Phi^{-1}()$.

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[12] Further information about Chauvenet' criterion can be found in *Experimentation and Uncertainty Analysis for Engineers*, 2nd Edition, H. W. Coleman and W. G. Steele, John Wiley & Sons, 1999.

$$L_n = \Phi^{-1}\left(\frac{1+P_n}{2}\right) \tag{2-13}$$

where

$$P_n = 1 - 1/2n .$$

Any points that lie outside $\bar{x} \pm L_n s_n$ are rejected.

2.2.7 <u>Sample Size Evaluation</u>. UncertaintyAnalyzer's Sample Size Evaluator, which is accessed from the Measurement Data Entry Worksheet or Error Source Uncertainty Worksheet, can be used to determine if the size of a sample of data is sufficient for obtaining an estimated sample mean that differs from the true (population) mean by less than or equal to some predetermined amount.

For example, let x_1, x_2, …, x_n represent repeated independent unbiased measurements from a distribution with mean μ and standard deviation σ. According to the law of large numbers, the sample average for these measurements, X, converges to μ in probability, and we can assume that X is a good estimate of μ if n is large.

The central limit theorem allows us to approximate how close X is to μ. Suppose that we want to find the probability $P(|X - \mu| < c)$ for some predetermined constant c. To estimate this probability we first standardize, using $E(X) = \mu$, and $\text{Var}(X) = \sigma_2 / n$:

$$P(|X - \mu| < c) = P(-c < X - \mu < c)$$

$$= P\left(\frac{-c}{\sigma/\sqrt{n}} < \frac{X - \mu}{\sigma/\sqrt{n}} < \frac{c}{\sigma/\sqrt{n}}\right)$$

$$\cong 2\Phi\left(\frac{c}{\sigma/\sqrt{n}}\right) - 1$$

where $\Phi()$ is the normal distribution function.

Equating this probability to a confidence level β for the condition $|\underline{x} - \mu| < c$, we have

$$\sqrt{n} \cong (\sigma/c)\Phi^{-1}\left(\frac{1+\beta}{2}\right).$$

In practice, we usually don't know the value of σ. Accordingly, we use the best available estimate. In many cases, this will be the sample standard deviation of the sample s, especially if n is large. To obtain a reliable estimate of the required value of n, then, we could take a large sample of, say fifty or more observations and estimate n as above.

Of course, this approach bears fruit if we wish to economize on taking future samples of measurements from the population. If not, the value of n obtained above is at least a quantity that deserves consideration.

2.2.8 <u>Estimated Uncertainty</u>. The estimated uncertainty due to random or repeatability error is equal to the sample standard deviation. The sample standard deviation is an estimate of the uncertainty in the uncertainty in a value drawn randomly from its population.

2.2.9 <u>Uncertainty in the Mean Value</u>. It is often desirable to estimate and report the uncertainty in the mean value of a random sample. In this case, the Use Uncertainty in the Mean option is selected on the Measurement Data Entry Worksheet or Error Source Worksheet.

The uncertainty in the mean value is the sample standard deviation divided by the square root of the sample size.

$$s_{\bar{x}} = \frac{s}{\sqrt{n}} \tag{2-14}$$

where

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

and

n = sample size

x_i = ith measured value

$\bar{x}$ = sample mean value

s = sample standard uncertainty (i.e., standard deviation)

The standard uncertainty in the mean is computed somewhat differently for the Sampled Cells, Sampled Mean Values and Mixed data entry options.

2.2.9.1 <u>Sampled Cells Option</u>. When this data entry option has been selected, the standard uncertainty in the mean value is computed as

$$S_{\bar{X}} = \frac{S \sqrt{\sum_{i=1}^{k} n_i^2}}{n} \tag{2-15}$$

where

n = total number of measured values
n_i = sample size for the ith sampling cell
k = number of sampled cells
s = standard uncertainty for all measured values

2.2.9.2 <u>Sampled Mean Values Option</u>. When mean values are entered into the Measurement Results table, the standard uncertainty in the mean value is computed as

$$S_{\bar{X}} = \frac{1}{n} \sqrt{\sum_{i=1}^{k} n_i s_i^2} \tag{2-16}$$

where

n = total number of measurements (i.e., cumulative of all sample sizes)
k = number of samples (i.e., number of mean values entered)
n_i = sample size for the ith sample
s_i = standard deviation of ith sample

2.2.9.3 <u>Mixed Option</u>. With this option, we use both the prior standard uncertainty estimate s_p together with the sample size n of the current sample. The standard uncertainty in the mean value is given by

$$S_{\bar{X}} = \frac{s_p}{\sqrt{n}},$$ (2-17)

where

        n = sample size of the current sample

        s_p = standard deviation of the prior sample.

If we wish to compute confidence limits around the sample mean for a confidence level of $(1 - \alpha) \times 100\%$, we employ the expression

$$\bar{x} \pm_{\alpha/2,v} \frac{s_p}{\sqrt{n}}$$ (2-18)

where

$$v = n_p - 1.$$

and $t_{\alpha/2,v}$ is the Student's t-statistic for a significance level of $\alpha / 2$ and v degrees of freedom.

CHAPTER 3

TYPE B UNCERTAINTY ESTIMATION

3.0 General

Uncertainty estimates for measurement process errors resulting from parameter bias, display resolution, operator bias, computation, stress response and environmental factors are typically determined heuristically via containment limits and containment probabilities.

This chapter discusses the concepts and methods used to estimate Type B uncertainties.[13] In particular, the methodology will focus on uncertainty estimation for the following measurement process errors:

- Measurement or Parameter Bias
- Resolution Error
- Digital Sampling Error
- Computation Error
- Operator Bias
- Environmental Factors Error
- Stress Response Error

The error distributions used for Type B uncertainty estimates are also discussed, along with guidelines for choosing appropriate distributions.

3.1 Definitions

3.1.1 <u>% Confidence</u>. See Confidence Level.

3.1.2 <u>% of Full Scale</u>. The contribution of a parameter's tolerance limit equal to a stated percentage of the parameter's full scale.

3.1.3 <u>% in-Tolerance</u>. See Percent In-Tolerance.

3.1.4 <u>% of Nominal</u>. The contribution of a parameter's tolerance limit equal to a stated percentage of the parameter's nominal value.

3.1.5 <u>% of Range</u>. The contribution of a parameter's tolerance limit equal to percentage of the parameter's range.

3.1.6 <u>Ancillary Parameters</u>. Parameters, other than the subject and measuring parameters, that participate in a measurement or indirectly influence the subject parameter or measuring parameter bias. An example of an ancillary parameter would be an environmental factor such as ambient temperature that is measured and used to apply an environmental correction to the subject or measuring parameter values. Sometimes referred to as *influence quantities*.

3.1.7 <u>Asymmetric Tolerances</u>. Two-sided tolerance limits in which the magnitudes of the upper and lower limits are unequal.

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[13] The methodology presented herein was developed by Dr. H. Castrup of Integrated Sciences Group (see References).

3.1.8 Bandwidth. The range of frequencies that comprise the sampled signal.

3.1.9 Bias. A systematic discrepancy between an indicated or declared value of an artifact and its true value.

3.1.10 Bias or Error Distribution. A statistical distribution that describes the relative frequency of occurrence of values of a measurement error.

3.1.11 Bias Offset or Offset from Nominal. The stated offset from nominal of a calibrated artifact.

3.1.12 Bias Uncertainty. The uncertainty in the bias of a parameter or artifact.

3.1.13 Confidence Level. The probability that a set of error limits or containment limits will contain errors for a given error source.

3.1.14 Confidence Limits. Limits that bound errors for a given error source with a specified probability or "confidence."

3.1.15 Containment Limits. Limits that are specified to contain either a parameter value, deviations from the nominal parameter value, or errors in the measurement of the parameter value.

3.1.16 Containment Probability. The probability that a parameter value or errors in the measurement of this value will lie within specified containment limits.

3.1.17 Digital Sampling. A process in which signal amplitudes are periodically sampled and converted to digital numbers.

3.1.18 Digital Sampling Error. The error resulting from analog to digital (A/D) conversion and subsequent digital to analog (D/A) conversion. Sources of digital sampling error include: sampling rate error, quantization error (sampling full scale and significant bits), aperture time error, impulse response error, sampling noise error, sampling noise, sensing error and hysteresis error.

3.1.19 Error Limits. Bounding values that are expected to contain the error from a given source with some specified level of probability or confidence.

3.1.20 Expanded Uncertainty. A multiple of the standard uncertainty reflecting either a specified confidence level or arbitrary coverage factor.

3.1.21 Fixed Limit. A tolerance limit for a parameter that does not depend on the nominal or other values associated with the parameter.

3.1.22 Full Scale. An identifying reference value for readings or outputs for a given parameter. Often the highest value in a range.

3.1.23 Heuristic. Pertaining to the use of general knowledge gained by experience, sometimes expressed as "using a rule-of-thumb."

3.1.24 Hysteresis. The resistance of a variable to a change in stimulus.

3.1.25 Hysteresis Error. The residual signal in a sampling event left over from the previous sampling event.

3.1.26 Impulse Response. The response of a sampling sensor to an instantaneous change in input.

3.1.27 <u>Impulse Response Error</u>. The error due to the finite time required for a sampling sensor to respond to a sampled signal. In UncertaintyAnalyzer, impulse response error includes hysteresis error.

3.1.28 <u>Influence Quantity</u>. See Ancillary Parameter.

3.1.29 <u>Interaction Coefficient</u>. A number that converts the value of an environment or ancillary parameter to a change exerted by the parameter on the bias of a subject parameter or measuring parameter. In error analysis, the interaction coefficient is the partial derivative of the measured variable with respect to the environment parameter.

3.1.30 <u>In-tolerance Probability</u>. The probability that a parameter value or the error in the value is contained within its specified tolerance limits at the time of measurement.

3.1.31 <u>Measurement Bias</u>. An error that persists from measurement to measurement during a measurement session.

3.1.32 <u>Measuring and Test Equipment</u>. The device or artifact featuring the Measuring Parameter.

3.1.33 <u>Nominal Value</u>. The designated or published value of an artifact or parameter. It may also sometimes refer to the mode value of an artifact or parameter.

3.1.34 <u>Number of p-n Junctions</u>. The number of positive to negative junctions in the sampling sensor. For example, if the sensor is a transistor, the number of junctions is 2.

3.1.35 <u>Operator Bias</u>. Error due to the perception or influence of a human operator or other agency. Operator bias is often referred to as a source of reproducibility error.

3.1.36 <u>Operating Temperature</u>. The temperature at which the sampling sensor is operated.

3.1.37 <u>Output Resistance</u>. The real part of the impedance of the sampling sensor.

3.1.38 <u>Parameter</u>. A discrete function of a device this is assigned a tolerance specification. For example, the DC voltage reading would be a parameter of a digital multimeter.

3.1.39 <u>Parameter Bias</u>. A systematic deviation of a parameter value from its nominal or indicated value.

3.1.40 <u>Parameter Drift</u>. A systematic long-term variation resulting from environmental conditions encountered during usage, calibration or storage.

3.1.41 <u>Percent In-Tolerance</u>. The probability, expressed in percent, that a parameter value or the error in the value is contained within its specified tolerance limits at the time of measurement.

3.1.42 <u>Precision</u>. Precision corresponds to how many places past the decimal point we can express a measurement result. Although higher precision does not necessarily mean higher accuracy, the lack of precision in a measurement is a source of measurement error.

31.43 <u>Quantization</u>. The process of converting a continuous, analog input or signal to a digitized code comprised of 0's and 1's.

3.1.44 <u>Quantization Error</u>. Error due to the granularity of resolution in quantizing a sampled signal.

3.1.45 Range. In a parameter specification, a designated range of values for which specified tolerances apply. In a calibration or test procedure, a setting or designation for a set of measurements.

3.1.46 Reference Standard. An artifact used as a measurement reference whose value and uncertainty have been determined by calibration.

3.1.47 Reproducibility. The closeness of the agreement between the results of successive measurements of the value of a parameter carried out under changed measurement conditions. The changed conditions may include: principle of measurement, method of measurement, observer, measuring instrument(s), reference standard, location, conditions of use, time.

3.1.48 Resolution. The smallest discernible value indicated by a measuring parameter or subject parameter.

3.1.49 Resolution Error. The error due to the finiteness of the precision of a measurement.

3.1.50 Response Time Constant. The time required for the output of a sampling sensor to make the change from an initial value to 63.2% (or 1-1/e) of the final steady-state value resulting from a step-change in the input.

3.1.51 RMS Current. The root mean square value of the current through the sampling sensor.

3.1.52 Sampling Aperture. The time over which an individual sample is taken. See also Aperture Time.

3.1.53 Sampling Full Scale. The maximum peak-to-peak amplitude of a sampled signal.

3.1.54 Sampling Noise. Noise present in the sampling sensor. In electronic devices, noise may include thermal noise, shot noise, stray emf signals and so on. In mechanical and dimensional devices, noise may include vibration, temperature fluctuations, etc.

3.1.55 Sampling Rate. The rate at which an input signal is sampled.

3.1.56 Significant Bits. The number of bits used in quantizing a sampled signal.

3.1.57 Single-sided Tolerance Limit. A single-sided tolerance limit in which only the lower or upper limit is specified.

3.1.58 Stress Limits. Limits that are expected to contain stresses that may be encountered from a given source.

3.1.59 Stress Response Coefficient. A coefficient indicating the response of a parameter to a stimulus, such as stress, expressed in units appropriate for the source of the stimulus.

3.1.60 Stress Response Error. The error in a parameter value due to applied stress.

3.1.61 Symmetric Tolerances. Two-sided tolerance limits in which the magnitudes of the upper and lower limits are equal.

3.1.62 Tolerance Limits. Limits that bound acceptable parameter values.

3.1.63 Toleranced Parameter. A parameter with a set of specified tolerances or limits that bound its value or the error in this value.

3.1.64 <u>Two-sided Tolerance Limits</u>. Upper and lower limits that bound acceptable parameter values. Stated upper and lower limits that contain biases in parameter value with some specified level of confidence. Two-sided tolerance limits can be symmetric or asymmetric.

3.1.65 <u>User Defined Error</u>. User specified error source other than the measurement process error sources incorporated into UncertaintyAnalyzer.

3.2 Subject and Measuring Parameter Bias

The uncertainty in the systematic error or bias of a parameter is typically estimated heuristically via containment limits and containment probabilities. In some instances, as with reference standards, the stated parameter value and its associated uncertainty estimate may be supplied by a higher-level calibration laboratory. In this case, the parameter bias uncertainty might have been obtained from statistical analysis of the calibration data and associated measurement process errors.

In UncertaintyAnalyzer, parameter bias uncertainty is computed using the Subject or Measuring Parameter Bias Uncertainty Worksheet. Parameter bias uncertainty estimates can be developed for reference standards, toleranced parameters, or from information about expected containment limits. Parameter tolerance specifications can be entered as a Fixed limit, % of Nominal or Reading, % of Full Scale, and % of Range.

The specifications can be entered as single-sided, two-sided asymmetric or two-sided symmetric tolerances and can be combined linearly or by the root-sum-square (RSS) method. In addition, the built-in SpecMaster Worksheet, described in Section 3.10, provides a tool for determining tolerances for parameters with complex specifications.

The user can choose from an extensive list of distributions for describing the statistical characteristics of the parameter biases. Supported distributions in UncertaintyAnalyzer include: Normal, Lognormal, Exponential, Quadratic, Cosine, U-Shaped, Uniform (rectangular), Triangular, and Student's t. Plots of these distributions are displayed in the Parameter Bias Uncertainty Worksheet to assist in developing parameter bias uncertainty estimates.

The parameter bias uncertainty reported at the bottom of the Parameter Bias Uncertainty Worksheet is computed from option selections; tolerance limits, expanded uncertainty or containment limits; and the confidence level or in-tolerance probability at the time of measurement.

3.3 Resolution Error

The uncertainty resulting from the finite resolution of a reading or output is usually determined from containment limits, which are based on the resolution of the device, and an estimated containment probability.

In UncertaintyAnalyzer, resolution uncertainty is computed using the Parameter Resolution Error Worksheet. The distribution for resolution error depends on whether the reading or output is expressed in analog or digital format.

3.3.1 <u>Analog Display Resolution</u>. For analog displays, the resolution error is assumed to have a normal distribution. The resolution uncertainty is computed by setting the containment limits equal to the smallest increment of resolution and applying a containment probability that readings can be discerned within these limits.

3.3.2 Digital Display Resolution. For digital displays, the resolution error is assumed to have a uniform distribution. The containment limits are ± half the smallest displayed digit and the containment probability is 100%.

3.4 Operator Bias

Operator bias uncertainty resulting from error in the perception of a human operator may be determined statistically, but is usually estimated heuristically. The usual way is to link it to resolution uncertainty or some other aspect of a measurement that can influence operator perception. The underlying error distribution is typically normal, but in some cases, the uniform distribution may apply.

In UncertaintyAnalyzer, operator bias uncertainty is computed using the Operator Bias Uncertainty Worksheet. Operator bias uncertainties can be estimated for both the subject parameter and measuring parameter readings.

3.5 Digital Sampling Error

Digital sampling uncertainty estimation involves defining a representative "signal" to be sampled and specifying a sampling rate, a sampling aperture time, a quantization precision (sampling full scale and significant bits), an impulse response and hysteresis, a sampling noise level, and a sensor bias uncertainty. It also involves deciding on a model or methodology for eventual conversion of digitized data back to analog form. This multi-faceted analysis process is difficult without a structured template approach.

UncertaintyAnalyzer contains specially designed screens to evaluate digital sampling uncertainty. Both the analog-to-digital and digital-to-analog conversion processes are handled. Sampling full scale, significant bits, and aperture time are entered in the Digital Sampling Uncertainty Worksheet. The following drill-down analysis screens and worksheets can be accessed from the Digital Sampling Worksheet:

- Input Signal Characteristics Worksheet
- Sampling Impulse Response Worksheet
- Sampling Noise Worksheet
- Sampling Sensor Worksheet

3.5.1 Input Signal Characteristics Worksheet. The worksheet is used to estimate uncertainty due to sampling rate error. Because signal activity occurs between samples, the rate at which samples are collected can introduce error into the sampling process. The magnitude of this error depends on the signal being sampled, the conversion of the continuous analog signal to discrete digitized values (A-D), and on the conversion from digital to analog (D-A) following digital signal processing.

The input signal amplitude, amplitude units (e.g. V), and signal frequency units (e.g., Hz) are entered in this worksheet. The sampling rate or sampling frequency is also entered, along with the sampling units (e.g. kHz).

The input is defined from up to 12 components of the representative signal. Each component is expressed in terms of a signal amplitude, frequency, and relative phase that is expressed in degrees or radians. The signal components can be harmonics of a base frequency. A linear or quadratic spline model can be selected to represent the level of digital to analog conversion. A

plot of the signal and the digital sampling points can also be displayed.

3.5.2 Sampling Impulse Response Worksheet. This worksheet is used to estimate the uncertainty due to impulse response error. Impulse response error is the discrepancy between the instantaneous signal value and the sampled value resulting from the finite time required for the sampling sensor to respond to input stimuli.

The impulse response $a(t)$ is modeled using an exponential response function

$$a(t) = A_0 + (A - A_0)(1 - e^{-t/t_c}) \tag{3-1}$$

where

$\quad A_0$ = impulse response at time $t = 0$

$\quad A$ = instantaneous signal value

$\quad t_c$ = response time constant

The response time constant is the time required for the sensor to achieve the value

$$A_0 + (1 - e^{-1})(A - A_0) \tag{3-2}$$

At any given time t, within the aperture time, the error in the sensed value is

$$a(t) - A . \tag{3-3}$$

The shorter the sensor response time constant, the closer the value of $a(t)$ gets to the instantaneous signal value A. Consequently, the faster the sensor responds to signal changes, the smaller the impulse response error will be. Conversely, as the sensor response time constant approaches the aperture time, the impulse response error increases significantly.

3.5.2.1 Sampling Impulse Response Error. Assume that the response of a system to a sensed value V is exponential:

$$r(t) = V(1 - e^{-\lambda t}) \tag{3-4}$$

with error

$$\varepsilon(t) = V e^{-\lambda t} \tag{3-5}$$

The unknowns here are the exact time at which the response function is applied, the response parameter λ, and the amplitude of the value V. Since we're focusing on hysteresis uncertainty, we'll consider only the uncertainty due to the sampled time. We'll assume that V can be represented by the average signal value and that the uncertainty in λ is negligible. The impulse response uncertainty is, accordingly, given by

$$\sigma_r^2(t) = \left(\frac{\delta\varepsilon}{\delta\lambda}\right)^2 \sigma_\lambda^2 + \left(\frac{\delta\varepsilon}{\delta V}\right)^2 \sigma_V^2 + \left(\frac{\delta\varepsilon}{\delta t}\right)^2 \sigma_t^2$$

$$\cong \left(\frac{\delta\varepsilon}{\delta t}\right)^2 \sigma_t^2 \tag{3-6}$$

$$= V^2 \lambda^2 \sigma_t^2 e^{-2\lambda t}$$

and

$$\sigma_r(t) = V\lambda\sigma_t e^{-\lambda t}.$$

(3-7)

The uncertainty σ_t is the uncertainty in the location of the sampled point. If this point can be established with confidence limits, then the standard deviation of σ_t can be determined as usual. If the location of the sampled point is unknown, the sampling error is taken as an average over the aperture τ according to

$$\bar{\varepsilon}_r = \frac{V}{\tau}\int_0^\tau e^{-\lambda t}dt$$

(3-8)

Determining σ_r in these cases involves the usual definition of the variance

$$\begin{aligned}
\sigma_r{}^2 &= \frac{1}{\tau}\int_0^\tau \left[\varepsilon_r(t) - \bar{\varepsilon}_r\right]^2 dt \\
&= \frac{1}{\tau}\int_0^\tau \varepsilon_r{}^2(t)dt - \bar{\varepsilon}_r{}^2 \\
&= \frac{V^2}{2\lambda\tau}(1 - e^{-2\lambda\tau}) - \frac{V^2}{(\lambda\tau)^2}(1 - e^{-2\lambda\tau})^2 \\
&= \left(\frac{V}{\lambda\tau}\right)^2\left[\frac{\lambda\tau}{2}(1 - e^{-2\lambda\tau}) - (1 - e^{-2\lambda\tau})^2\right]
\end{aligned}$$

(3-9)

The parameter λ is obtained from the time required for the response to reach $1/e$ the input value:

$$\lambda = \frac{1}{\tau_e}$$

(3-10)

3.5.2.2 <u>Hysteresis Error</u>. Hysteresis error arises from residual values left over from previous samples. Represent this quantity by the variable V_h. If the sampling rate is v_s, then

$$V_h = V(1 - e^{-\lambda\tau})e^{-l(T-\tau)}$$

(3-11)

where

$$T = 1/v_s$$

(3-12)

The hysteresis error is the sampled value of this residual amount. Thus the hysteresis error is given by

$$\varepsilon_h(t) = V_h\left(1 - e^{-\lambda t}\right)$$

(3-13)

If the sample point t is unknown, we instead use the average error

$$\bar{\varepsilon}_h = V_h\left[1 - \frac{1}{\lambda\tau}\left(1 - e^{-\lambda t}\right)\right].$$

(3-14)

The hysteresis uncertainty can be obtained from the hysteresis error in the same way as the impulse response uncertainty was obtained. If the sample point is known, then

$$\sigma_h = \lambda V_h e^{-\lambda t} \sigma_t \tag{3-15}$$

whereas, if the sample point is unknown,

$$
\begin{aligned}
\sigma_h{}^2 &= \frac{1}{\tau} \int_0^\tau \varepsilon_h{}^2(t) - \overline{\varepsilon}_h \\[1mm]
&= V_h{}^2 \left[1 - \frac{2}{\lambda \tau}\left(1 - e^{-\lambda \tau}\right) + \frac{1}{2\lambda \tau}\left(1 - e^{-2\lambda \tau}\right) \right] - \overline{\varepsilon}_h{}^2 \\[1mm]
&= \frac{V_h{}^2}{\lambda \tau}\left[\frac{1}{2}\left(1 - e^{-2\lambda \tau}\right) - \frac{1}{\lambda \tau}\left(1 - e^{-\lambda \tau}\right)^2 \right]
\end{aligned}
\tag{3-16}
$$

3.5.3 <u>Sampling Noise Worksheet</u>. This worksheet is used to estimate the uncertainty in the sensed value of a sampled point due to noise. In electronic devices, noise may include thermal noise, shot noise, stray emf signals and so on. In mechanical and dimensional devices, noise may include vibration, temperature fluctuations, etc.

The Sampling Noise Worksheet allows the user to input information about electronic noise and other noise sources. The electronic noise portion of the worksheet provides a means for computing Thermal Noise and Shot Noise uncertainty estimates. Thermal noise is the noise due to random motion of current carriers (e.g., electrons) in a sampling sensor. Shot noise is the noise due to random fluctuations in the number of carriers in a semiconductor device.

The uncertainty in the sampled value due to thermal noise is estimated from user input values for the Bandwidth, Operating Temperature, and Output Resistance. The uncertainty in the sampled value due to shot noise is estimated from the RMS current and the number of sensor p-n junctions, along with the bandwidth and output resistance. The electronic signal to noise ratio is also computed as supplemental information. It can also be input by the user, if desired.

This other noise portion of the Sampling Noise Worksheet allows the user to input error limits and associated confidence levels for other sources of signal noise. The error limits for a given noise source should be input in amplitude units associated with the sample signal that is defined in the Input Signal Characteristics Worksheet. If the error limits are input in tolerance units or units from a different measurement area, then the appropriate conversion factor must be entered into the Coefficient data field. The coefficient data field can also be used to input a multiplying factor if the noise source undergoes attenuation or amplification by some external factor.

3.5.4 <u>Sampling Sensor Worksheet</u>. This worksheet is used to estimate the uncertainty in the sampled value due to the error in the sampling sensor. This worksheet allows the user to input sensor tolerance limits and associated confidence level. If a confidence level of 100% is entered, then the sensor error is assumed to be uniformly distributed. Otherwise, a normal distribution is assumed for the sensor error.

3.6 Computation Error

Computation uncertainty can result from round-off or truncation error and the application of empirical equations or the results of curve-fit or regression analysis to compute parameter values. Uncertainty resulting from round-off error and the use of empirical equations is usually determined heuristically. The uniform distribution is applicable for round-off or computer

truncation error. The triangular distribution is appropriate for describing errors resulting from linear interpolation. UncertaintyAnalyzer's Computation Error Uncertainty Worksheet provides a tool for estimating computation uncertainty.

3.6.1 <u>Round-off of Truncation</u>. Since computer or calculator precision is limited, each step of a computation generates uncertainty. These uncertainties propagate through the computation, emerging as part of the result. Uncertainty resulting from round-off error is usually determined heuristically. The uniform distribution is applicable for round-off or computer truncation error.

In most cases, the use of double-precision arithmetic can reduce computation error uncertainty. However, uncertainty due to round-off or computer truncation can become serious for quantities based on a large number of calculations, such as in iterative processes, calculations involving matrix products or inverses, calculations involving trigonometric or lognormal functions, etc.

Computation errors contribute to the uncertainty in the resulting value in varying degrees. For example, if the calculated value is the sum of two quantities, the computation error uncertainty is the root sum square (RSS) of the round-off uncertainties for each quantity. However, if the calculated value is the product of two quantities that are multiplied, the computation error uncertainty is the RSS of the second quantity times the round-off error of the first quantity and the first quantity times the round-off error of the second quantity.

For illustration, we will evaluate the computation error for the temperature measurement of a substance using a type J thermocouple that gives output values in °C. The temperature output of the thermocouple is actually a computed value based on thermocouple reference tables that convert millivolt output to °C.

The conversion from millivolt output to °C is done via some external device that is connected to the thermocouple leads. The conversion is either obtained from interpolation of tabulated values or from an equation that "best fits" the tabulated data. In any event, some computation error is introduced into the temperature output value.

In this example, we will assume that the millivolt (mV) output of the thermocouple is converted to °C by interpolating between tabulated values. In this case, the interpolation equation would be

$$T_{output} = mV_{output}\ \frac{T_1 - T_2}{mV_1 - mV_2} \tag{3-17}$$

A number of computational error sources can be entered in the Round-off error source table of the Computation Error Uncertainty Worksheet. In the thermocouple example, each of the quantities, mV_{output}, T_1, T_2, mV_1 and mV_2, in equation (3-17) represents a computation error source and must be listed as such in the table. For instance, we could enter the following error source descriptions and values:

    Millivolt Output = 1.254 mV
    Temperature Reference1 = 24 °C
    Temperature Reference2 = 25 °C
    Millivolt Reference1 = 1.2253 mV
    Millivolt Reference2 = 1.2770 mV

When entering values in the **Typical Value** data fields, care must be taken to ensure that the

resulting computation uncertainty for the error source is in the subject parameter nominal or tolerance units that have been selected for reporting the uncertainty estimates.

In the **Approx. No. Calcs.** data field, the user inputs an estimated number of calculations involved in obtaining the computed or measured value of the error source. For the thermocouple example, the error source values are not actually computed, so we enter a value of 1 for each error source.

In the **Decimal Digits** data field, the user inputs the precision of the device that is making the calculations (i.e. number of digits to the right of the decimal point). In the thermocouple example, we input the number of digits to the right of the decimal place for each error source value. However, we are interpolating to obtain the temperature output and know that its value will fall between 24 °C and 25 °C. Accordingly, we will want to obtain an output temperature with a precision of at least 2 decimal places and need to assume a precision of at least 2 decimal places for the reference temperatures. The resulting decimal precision for the error sources are listed below.

Error Source Description	Typical Value	Decimal Digits
Millivolt Output	1.254	3
Temperature Reference1	24.00	2
Temperature Reference2	25.00	2
Millivolt Reference1	1.2253	4
Millivolt Reference2	1.2770	4

We must now determine the partial derivatives of equation (3-17) with respect to each of the computation error source quantities, mV_{output}, T_1, T_2, mV_1 and mV_2. The numerical values of the partial derivatives are computed and entered into the **Sensitivity Coefficient** data field.

The partial derivative of the output temperature equation with respect to mV_{output} is simply

$$\frac{\partial T_{output}}{\partial mV_{output}} = \frac{T_1 - T_2}{mV_1 - mV_2} \tag{3-18}$$

Entering the computed or measured values for T_1, T_2, mV_1 and mV_2 into equation (3-18) gives us the value of the Source Coefficient for the Millivolt Output error source.

$$\text{Sensitivity Coefficient for } mV_{output} = \frac{24 - 25}{1.2253 - 12770} = 19.3424$$

The value of 19.3424 is then entered into the Source Coefficient data field. The partial derivative of the output temperature equation with respect to T_1 is

$$\frac{\partial T_{output}}{\partial T_1} = \frac{mV_{output}}{mV_1 - mV_2} \tag{3-19}$$

and entering the computed or measured values for mV_{Vouput}, mV_1 and mV_2 into the above equation gives us the value of the Source Coefficient for the Temperature Reference1 error source.

$$T_1 \text{ Source Coefficient} = \frac{1.2540}{1.2253 - 1.2770} = -24.2553$$

Similarly, the partial derivative of the output temperature equation with respect to *T2* is

$$\frac{\partial T_{output}}{\partial T_2} = \frac{-mV_{output}}{mV_1 - mV_2} \tag{3-20}$$

and the Source Coefficient for the Temperature Reference2 error source is

$$Sensitivity\,Coefficient\,for\,T_2 = \frac{1.2540}{1.2253 - 1.2770} = 24.2553$$

The partial derivative of the output temperature equation with respect to mV_1 is

$$\frac{\partial T_{output}}{\partial mV_1} = -mV_{output}\frac{(T_1 - T_2)}{(mV_1 - mV_2)^2} \tag{3-21}$$

and the Source Coefficient for the Millivolt Reference1 error source is

$$Sensitivity\,Coefficient\,for\ mV_2 = -1.2450\frac{(24 - 25)}{(1.2253 - 1.2770)^2} = 469.17$$

Similarly, the partial derivative of the output temperature equation with respect to mV_2 is

$$\frac{\partial T_{output}}{\partial mV_2} = mV_{output}\frac{(T_1 - T_2)}{(mV_1 - mV_2)^2} \tag{3-22}$$

and the Source Coefficient for the Millivolt Reference2 error source is

$$Sensitivity\,Coefficient\,for\ T_2 = 1.2540\frac{(24 - 25)}{(1.2253 - 1.2770)^2} = -469.17$$

UncertaintyAnalyzer employs the uniform distribution for estimating uncertainty due to round-off or other computational error. The value displayed in the Computation Uncertainty data field is computed as follows:

1. The error limits are assumed to be equal to ± half of the last decimal place for the error source

$$error\,limits = \pm\left(0.5 \times 10^{-D}\right) \tag{3-23}$$

   where D is the number of decimal digits.

2. The error limits are multiplied by the square root of the number of calculations, *n*, and divided by the square root of 3 to obtain an estimate of the standard uncertainty for a uniformly distributed error.

$$standard\,uncertainty = \frac{\sqrt{n}\left(0.5 \times 10^{-D}\right)}{\sqrt{3}} \tag{3-24}$$

3. The standard uncertainty is then multiplied by the sensitivity coefficient to obtain an estimate of the computation uncertainty for the error source.

$$\text{Computation Uncertainty} = \text{Sensitivity Coefficient} = \frac{\sqrt{n}\left(0.5 \times 10^{-D}\right)}{\sqrt{3}} \qquad (3\text{-}25)$$

For the thermocouple example, the computation uncertainty for each error source is listed in the Round-Off data table as shown below.

Error Source Description	Computer or Measured Value	Approx. No. Calcs	Decimal Digits	Source Coefficient	Computation Uncertainty
Millivolt Output	1.254	1	3	19.342	0.0056
Temperature Reference1	24.00	1	2	-24.255	0.0700
Temperature Reference2	25.00	1	2	24.255	0.0700
Millivolt Reference1	1.2253	1	4	469.2	0.0135
Millivolt Reference2	1.2770	1	4	-469.2	0.0135

The bottom portion of the Computation Error Uncertainty Worksheet displays the total uncertainty due to computation error. The total computation uncertainty is obtained by combining the computation uncertainties for the individual error sources in a root-sum-square (RSS) manner.

For the thermocouple example, the total computation uncertainty is calculated to be 0.101 °C.

$$\begin{aligned}
\text{Total Computation Uncertainty} \quad &= \sqrt{(0.0056)^2 + 2(0.07)^2 + 2(0.0135)^2} \\
&= \sqrt{0.0102} \\
&= 0.101
\end{aligned}$$

3.6.2 <u>Empirical Equations</u>. In some instances, we may use an equation to predict the value of a variable y for any given values of one or more variables x_i. Sometimes, a physical law or principle connects the variables so that a mathematical equation can be used to express y as a function of the variables x_i. However, in complex measurement systems, there may be a more empirical relationship that must be estimated from experimental observation.

Empirical equations are defined by estimated coefficients whose values have uncertainty. UncertaintyAnalyzer's Empirical Equations tab of the Computation Error Uncertainty Screen provides a tool for estimating the total uncertainty resulting from the use of empirical equations. The equation and associated variables and coefficients are entered along with their typical value, error or containment limits and containment probability. The error distribution is selected from a drop down list for each variable (where appropriate). UncertaintyAnalyzer then uses this information to compute the standard uncertainty for each variable. Alternatively, the user can enter the standard uncertainty for each variable.

3.6.3 <u>Regression Analysis</u>. Regression analysis, a trend line is fit to the observed data. In UncertaintyAnalyzer's Regression Analysis Worksheet, users have the option of applying 1st, 2nd, or 3rd degree polynomials. The equations for the trend lines are

$$\hat{y} = \sum_{r=0}^{m} b_r x^r, \quad m = 1, 2, \text{ or } 3$$

where

$\hat{y}$ represents a predicted dependent variable value, and

x represents the corresponding independent variable value.

In the discussion to follow, the 3rd degree (m = 3) polynomial model is treated explicitly. Extension to 1st and 2nd degree models is straightforward.

The regression or curve-fit process begins by minimizing a quantity called the *residual sum of squares* with respect to the polynomial model coefficients b_r, $r = 0, 1, ... , m$. Next, the stage is set for predicting values of given values of x. Each predicted value of $\hat{y}$ is accompanied by an uncertainty. Estimating the uncertainty in a predicted value requires the development of a *variance-covariance matrix*.

3.6.3.1 <u>Residual Sum of Squares</u>. The regression analysis consists of finding a polynomial fit to the data. Sampled data are arranged as follows:

Independent Variable	Dependent Variable	Number of Sampled Values
x_1	y_1	n_1
x_2	y_2	n_2
.	.	.
.	.	.
.	.	.
x_k	y_k	n_k

Given the above, the residual sum of squares is written

$$RSS = \sum_{i=1}^{k} n_1 \left(y_i - \hat{y}_i\right)^2 \qquad (3\text{-}26)$$

where

y_i is the ith observed independent variable value,

n_i, is the number of values of y sampled at the point x_i, and

$\hat{y}_i$ is computed from the value x_i *using* the applicable polynomial.

The coefficients of the model are solved for by minimizing RSS.

3.6.3.2 Regression Model Coefficients. The solutions for the coefficients are estimated using matrices. The solutions for the coefficients are given by equation (3-27).

$$\mathbf{b} = \left(\mathbf{X'WX}\right)^{-1}\mathbf{X'WY} \qquad (3\text{-}27)$$

To illustrate, the matrices for the 3rd degree polynomial model are

$$\mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 \end{pmatrix} \qquad \mathbf{W} = \begin{pmatrix} n_1 & & & 0 \\ & n_2 & & \\ & & \ddots & \\ 0 & & & n_k \end{pmatrix} \qquad \text{and } \mathbf{Y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

and $\mathbf{X'}$ is the transpose of $\mathbf{X}$. Matrices for the 1st and 2nd degree polynomial models can be inferred from the above.

3.6.3.3 <u>Variance-Covariance Matrix</u>. The variance-covariance matrix of the regression coefficients vector **b** is given by

$$V = \mathrm{var}(\mathbf{b})$$
$$= (\mathbf{X'WX})^{-1} s^2 \tag{3-28}$$

The quantity s^2 is an estimate of the variance in the independent variable computed by

$$s^2 = \frac{RSS}{n-p}, \tag{3-29}$$

where

$$n = \sum_{i=1}^{k} n_i, \tag{}$$

and

$$p = m+1. \tag{}$$

.

For the 3$^{\mathrm{rd}}$ degree model, the variance-covariance matrix is

$$\mathbf{V} = \begin{bmatrix} \mathrm{var}(b_0) & \mathrm{cov}(b_0,b_1) & \mathrm{cov}(b_0,b_2) & \mathrm{cov}(b_0,b_3) \\ \mathrm{cov}(b_1,b_0) & \mathrm{var}(b_1) & \mathrm{cov}(b_1,b_2) & \mathrm{cov}(b_1,b_3) \\ \mathrm{cov}(b_2,b_0) & \mathrm{cov}(b_2,b_1) & \mathrm{var}(b_2) & \mathrm{cov}(b_2,b_3) \\ \mathrm{cov}(b_3,b_0) & \mathrm{cov}(b_3,b_1) & \mathrm{cov}(b_3,b_2) & \mathrm{var}(b_3) \end{bmatrix} \tag{3-30}$$

where

$$\mathrm{cov}(b_i,b_j) = \mathrm{cov}(b_j,b_i) \quad i = 0,1,2,3 \quad j = 0,1,2,3 \quad i \neq j$$

3.6.3.4 <u>Predicted Values and Uncertainties</u>. A value $\hat{y}$ is computed for a given value of x according to

$$\hat{y}(x) = \mathbf{x'b} \tag{3-31}$$

where, for the 3$^{\mathrm{rd}}$ degree polynomial model,

$$\mathbf{x'} = \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \tag{3-32}$$

The variance in is estimated according to

$$\mathrm{var}[\hat{y}(x)] = \mathbf{x'}(\mathbf{X'WX})^{-1}\mathbf{x}s^2$$
$$= \mathbf{x'Vx} \tag{3-33}$$

where **x** is the transpose of **x'**.

The quantity

$$s_{\bar{y}/x} = \text{var}[\hat{y}(x)]$$
$$= \mathbf{x'Vx}$$

(3-34)

is the standard error in a predicted *mean* value of y for a given value of x.

Since the actual value of y varies about the true mean value with a variance estimated by s^2, the variance of an *individual* predicted value is estimated by

$$\text{var}[\hat{y}(x)] = s^2 + \mathbf{x'(X'WX)}^{-1}\mathbf{x}s^2$$

(3-35)

The quantity

$$s_{y/x} = \sqrt{s^2 + \text{var}[\hat{y}(x)]}$$
$$= \sqrt{1 + \mathbf{x'(X'WX)}^{-1}\mathbf{x}} \; s^2$$

(3-36)

is called the *standard error in the forecast*. It represents the uncertainty in a projected individual value of y, given a value of x.

3.6.3.5 <u>Regression Model Testing</u>. The method of regression model testing discussed in this section applies to m-order polynomial cases in which n sampled values are grouped into k cells, each corresponding to a sampled independent variable x_i, $i=1,2,\ldots,k$. We require that $p < k < n$.

The inherent scatter in the sampled dependent variable values or *pure error* is quantified by the pure error variance given by

$$s_e^2 = \frac{1}{v_e} \sum_{i=1}^{k} n_i (y_i - \bar{y})^2$$

(3-37)

where

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} n_i y_i$$

(3-38)

and

$$v_e = n - 1 .$$

(3-39)

The quantity

$$\sum_{i=1}^{k} n_i \left(y_{ij} - \bar{y}_i \right)^2$$

(3-40)

is the sum of squares about the mean for k-sample cases. For these cases, the variance due to lack of fit is written

$$s_L^2 = \frac{1}{V_L} \sum_{i=1}^{k} n_i \left(\hat{y}_i - \bar{y} \right)^2$$

(3-41)

where

$$V_L = v_e - (n - p)$$
$$= p - 1 \qquad (3\text{-}42)$$
$$= m$$

The quantity

$$\sum_{i=1}^{k} n_i (\hat{y}_i - \bar{y})^2 \qquad (3\text{-}43)$$

is called the *lack of fit sum of squares* for k-sample cases.

If the lack of fit sum of squares is close to or less than the sum of squares about the mean, we suspect that we have a good model.

The confidence for rejecting the model, involves using two χ^2 variables $V_L s_L^2 / \sigma^2$ and $v_e s_e^2 / \sigma^2$ and forming a test variable F, given by

$$F = \frac{s_L^2}{s_e^2} \qquad (3\text{-}44)$$

This quantity is an F-distributed variable with v_L and v_e degrees of freedom. The variable F is the ratio of the variance due to scatter about regression to the variance due to pure error. If the former is smaller than the latter, then we can say that we have a viable model. That is, if we have a small value of F, the confidence for rejecting the model should be low. Accordingly, we use F as a test statistic and compute

$$\alpha = \int_0^F f_F(t; v_s, v_{s_L}) dt \qquad (3\text{-}45)$$

where α is the rejection confidence.

The best model is considered to be the model with the lowest rejection confidence of all available models. In Range Fit, the available models are polynomial models from first to third degree. Consequently, the best model is obtained using the following algorithm:

$C_{min} = 10^{307}$
$r = 0$
For m = 1 to 3
    Fit model $y = b_0 + ... + b_m x$
    Compute the rejection confidence C_m
    If $C_m < C_{min}$ then
        $C_{min} = C_m$
        $r = m$
    End If
Next
Select model where $m = r$.

3.6.3.6 <u>Testing for Trend Significance</u>. This section discusses testing whether the observed data follow a significant trend. For this, a 1st degree regression fit is performed. In 1st degree fits, the regression model is

$$\hat{y} = b_0 + b_1 x \qquad (3.46)$$

Testing for significance of a trend consists of testing the magnitude of b_1. If the slope b_1 is sufficiently small, it cannot be claimed that a significant trend exists. Fortunately, the significance of a trend can be tested statistically. The test statistic to be used is

$$t = \frac{b_1}{\sqrt{\text{var}(b_1)}} \qquad (3\text{-}47)$$

where

$$\text{var}(b1) = \frac{s^2}{\sum_{i=1}^{k} n_i (x_i - \bar{x})^2} \qquad (3\text{-}48)$$

The applicable degrees of freedom is

$$v = n - 2 .$$

Given a significance level α, then if $t < t_{\alpha,v}$, the slope is not considered significant.

Nonlinearity is defined as the error due to deviation of values $y(x)$ from a linear relationship $f(x) = a + bx$, where a and b are constants. Specifically,

$$\text{Nonlinearity} = \sqrt{\frac{1}{q} \sum_{i=1}^{q} [y(x_i) - (a + bx_i)]^2} \qquad (3\text{-}49)$$

where q is a number of values to be determined by the method of nonlinearity computation.

The constants a and b are obtained as follows: Let x_{lower} and x_{upper} be lower and upper points for a range of values and let y_{lower} and y_{upper} be corresponding dependent variable values. Then, for the range of values in question,

$$a = x_{lower} \quad \text{and} \quad b = \frac{y_{upper} - y_{lower}}{x_{upper} - x_{lower}}$$

UncertaintyAnalyzer computes nonlinearity from two perspectives. One is the error due to deviations from observed values. The other is the error due to deviations from projected values.

In the case of deviation from observed values, $q = k$ and $y(x_i) = y_i$, $i = 1, 2, \dots , k$. Accordingly,

$$\text{Nonlinearity} = \sqrt{\frac{1}{k} \sum_{i=1}^{k} [y(x_i) - (a + bx_i)]^2} \qquad (3\text{-}50)$$

In the case of deviation from projected values, $y(x) = \hat{y}(x)$ where $\hat{y}(x)$ is a projected value at x, computed using the regression model. The number q can be any number that is sufficient to obtain a nonlinearity estimate. In Range Fit, $q = 50$.

$$\text{Nonlinearity} = = \sqrt{\frac{1}{50} \sum_{i=1}^{50} [y(x_i) - (a + bx_i)]^2} \qquad (3\text{-}51)$$

In this expression

$$x_i = x_{lower} + (i-1)\Delta, \tag{3-52}$$

where

$$\Delta = \frac{x_{upper} - x_{lower}}{q-1} = \frac{x_{upper} - x_{lower}}{q-1} \tag{3-53}$$

3.7 Environmental Factors Error

Environmental factors uncertainty is typically estimated heuristically in a three-step process:

1. Estimate the uncertainty in each environmental or ancillary process error.

2. Multiply each environmental or ancillary uncertainty by its respective interaction coefficient.

3. Combine the environmental or ancillary uncertainties, accounting for all correlations.

The interaction coefficient relates an environmental or ancillary factor to an error source. For example, if the error source under consideration is the measurement of length and the environmental factor is temperature, then the interaction coefficient is the thermal expansion coefficient for length.

UncertaintyAnalyzer contains specially designed screens to evaluate uncertainty resulting from environmental factors error. The Environmental Factors Uncertainty Worksheet contains an Environment Parameter Table that allows up to 10 environment parameter error sources for estimating the total measurement uncertainty due to environmental factors. The data for each environmental error source can be input directly into the Environment Parameters table or the corresponding Tolerance Worksheet, which is activated by clicking the ID Number button to the left of the environment parameter name.

The drill-down Tolerance Worksheet provides tool for estimating two types of uncertainties: 1) uncertainty in an environmental/ancillary correction or 2) uncertainty due environmental/ancillary variations. The worksheet entry fields change slightly depending upon the uncertainty analysis option that is chosen.

UncertaintyAnalyzer assumes that measurement uncertainties due to environmental factors are normally distributed unless the associated degrees of freedom are less than infinite or the confidence level is 100%. If the degrees of freedom are less than infinite and the confidence level is less than 100%, the student's t distribution is used. If the confidence level is 100%, then the uniform distribution is used and the degrees of freedom are assumed to be infinite.

3.8 Stress Response Error

Stress response uncertainty resulting from shipping and handling is usually estimated heuristically. The method of determination involves attempting to estimate the uncertainty in the stresses encountered and multiplying this uncertainty by a stress response coefficient (i.e., the response of the parameter value to stress).

Stress response uncertainty should be considered, for example, when an item or device has been calibrated in an external laboratory to account for any expected increase in the uncertainty

reported on the calibration certificate due to stresses incurred during transport back to the end-user. If a calibration laboratory already includes stress response uncertainty, it should be indicated in the uncertainty budget (i.e., the itemized list of error sources).

In UncertaintyAnalyzer, the Stress Response Uncertainty Worksheet provides for up to 30 stress response error sources in estimating the total uncertainty in the value of the subject parameter due to shipping and handling stress.

UncertaintyAnalyzer assumes that uncertainties due to shipping and handling stresses are normally distributed unless the degrees of freedom are less than infinite or the confidence level is 100%. If the degrees of freedom are less than infinite and the confidence level is less than 100%, the student's t distribution is used. If the confidence level is 100%, then the uniform distribution is used and the degrees of freedom are assumed to be infinite.

3.9 User Defined Errors

UncertaintyAnalyzer's Error Source Worksheet provides a tool for developing Type A, Type B and Type A,B uncertainty estimates for other, user specified error sources.

Type A uncertainty estimates for user defined error sources can be estimated statistically from measured values or deviations from nominal that are entered in the Type A Uncertainty data entry table. Entering values in the table results in the following statistics to be computed:

- Mean or average of the measured values.
- Average or mean of the measured deviations from nominal.
- Sample size.
- Standard uncertainty or deviation of the data sample.
- Standard uncertainty in the mean value.

Uncertainties for user defined error sources can be estimated heuristically from bounding values, or ± limits, that are expected to contain the error with some specified probability or confidence level. If appropriate, a coverage factor can be entered instead of a confidence level. The ± limits can be specified as a fixed tolerance limit, % of Nominal, % of Ref 1, % of Ref 2, % of Ref 3 or any linear or root-sum-square combination of these quantities.

In UncertaintyAnalyzer's Error Source Worksheet, Type B uncertainty analysis errors are typically assumed to be normally distributed. However, in some cases, the Normal distribution may not apply. If so, then the appropriate distribution can be selected from a drop-down Distribution List that contains the Normal, Quadratic, Cosine, U-Shaped, Uniform (Rectangular), Triangular, and Student's t.

UncertaintyAnalyzer assumes infinite degrees of freedom for the Type B uncertainty estimate unless otherwise specified. Since, the degrees of freedom variable quantifies the amount of knowledge available for making the uncertainty estimate, an infinite degrees of freedom signifies complete certainty, i.e., zero uncertainty.

In most cases, it would not be realistic to assume infinite knowledge about the uncertainty. The Type B Degrees of Freedom Calculator, which is accessed by clicking the Degrees of Freedom button, can be used to provide additional information about the uncertainty in the ± limits and associated confidence level.

3.10 SpecMaster

The performance characteristics of test and measurement equipment can drift when subjected to long-term exposure environmental conditions during storage or use. Parameter drift is common in electronic and microwave equipment. Consequently, some manufacturers may report equipment specifications that vary with time and/or environmental conditions.

UncertaintyAnalyzer's SpecMaster worksheet provides a tool for developing tolerance limits for parameters with complicated specifications. SpecMaster is accessed from the Parameter Bias Uncertainty Worksheets. Equations or algorithms for computing tolerance limits are entered in the designated area of the SpecMaster Worksheet and the specification data are entered in the table in the lower part of the worksheet.

Tolerance algorithms are developed using a simple but powerful scripting language called VBScript. A fairly comprehensive guide to VBScript is contained in a Help file accessed from the SpecMaster Help menu. Available mathematical functions are listed in the Math Functions section of the Help file. The Math Functions list contains functions that are intrinsic to VBScript and those that have been added for use in UncertaintyAnalyzer.

3.11 Type B Degrees of Freedom Calculator

As previously discussed in Chapter 1, if a Type B estimate is obtained solely from containment limits and containment probabilities, then the degrees of freedom is usually taken to be infinite. For example, if the measurement error is normally distributed, the uncertainty is computed from the containment limits, $\pm L$, the inverse normal distribution function, $\Phi^{-1}()$, and the containment probability, $p = C/100$.

$$u = \frac{L}{\Phi^{-1}\left(\frac{1p}{2}\right)} \tag{3-54}$$

However, if there is an uncertainty in the containment limits (e.g., $\pm L \pm \Delta L$) or the containment probability (e.g., $\pm p \pm \Delta p$), then it is important that the degrees of freedom reflect this "fuzziness" or lack of knowledge in our estimate.

UncertaintyAnalyzer's built-in Type B Degrees of Freedom Calculator is a tool that allows the user to provide additional information about the uncertainty in the $\pm$ limits and associated confidence level used to make Type B uncertainty estimates. This information is used to estimate the uncertainty and the associated degrees of freedom. The Type B Degrees of Freedom Calculator can be accessed from all screens and worksheets that deal with the estimation of Type B uncertainties.

3.11.1 Methodology. The approach used to estimate the degrees of freedom for Type B estimates begins with the relation proposed in the ISO GUM.

$$v \approx \frac{1}{2} \frac{u^2(x)}{\sigma^2[u(x)]} \tag{3-55}$$

The method for computing the variance in the uncertainty, $\sigma^2[u(\mathrm{x})]$, is outlined in the following steps:

1. We generalize the equation for the Type B uncertainty estimate as

$$u_B = \frac{L}{\varphi(p)} \tag{3-56}$$

where L in the containment limit, p is the containment probability, and $\varphi(p)$ is defined as

$$\varphi(p) = \Phi^{-1}[(1+p)/2] \tag{3-57}$$

and the function $\Phi^{-1}[\cdot]$ is the inverse normal distribution function.

2. The error in the uncertainty, u_B, due to errors in L and p is estimated using a Taylor Series expansion.

$$
\begin{aligned}
\varepsilon_{u_B} &= \left(\frac{\partial u_B}{\partial L}\right)\varepsilon_L + \left(\frac{\partial u_B}{\partial p}\right)\varepsilon_p \\
&= \left(\frac{\partial u_B}{\partial L}\right)\varepsilon_L + \left(\frac{\partial u_B}{\partial \varphi}\right)\frac{d\varphi}{dp}\varepsilon_p \\
&= \frac{\varepsilon_L}{\varphi} - \frac{L}{\varphi^2}\frac{d\varphi}{dp}\varepsilon_p
\end{aligned}
\tag{3-58}
$$

where ε_L and ε_L are errors in L and p, respectively.

3. Assuming statistical independence between these errors, the variance in u_B follows directly:

$$
\begin{aligned}
\sigma^2(u_B) &= \operatorname{var}(\varepsilon_{u_b}) \\
&= \frac{1}{\varphi^2}\operatorname{var}(\varepsilon_L) + \frac{L^2}{\varphi^4}\left(\frac{d\varphi}{dp}\right)^2 \operatorname{var}(\varepsilon_p)
\end{aligned}
\tag{3-59}
$$

By definition, the uncertainty, u_x, of a quantity x is equal to the square root of the variance in the error in x.

$$u_x = \sqrt{\operatorname{var}(\varepsilon_x)}$$

Therefore, the variance in ε_L and ε_p can be expressed as

$$
\begin{aligned}
\sqrt{\operatorname{var}(\varepsilon_L)} &= u_L^2 \\
\sqrt{\operatorname{var}(\varepsilon_p)} &= u_p^2
\end{aligned}
$$

Equation (3-59) can then be expressed as

$$\sigma^2(u_B) = \text{var}(\varepsilon_{u_b})$$

$$= \frac{u_L^{\,2}}{\varphi^2} + \frac{L^2}{\varphi^4}\left(\frac{d\varphi}{dp}\right)^2 u_p^{\,2} \tag{3-60}$$

Dividing equation (3-60) by the square of equation (3-56), we get

$$\frac{\sigma_{2(u_B)}^2}{u_B^{\,2}} = \frac{u_L^{\,2}}{L^2} + \frac{1}{\varphi^2}\left(\frac{d\varphi}{dp}\right)^2 u_p^{\,2} \tag{3-61}$$

The derivative in equation (3-61) is obtained from equation (3-57). We first establish that

$$\frac{1+p}{2} = \Phi[\varphi]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\varphi} e^{-\zeta^2/2}d\zeta \tag{3-62}$$

where $\Phi[\cdot]$ is the probability density function for the normal distribution.

We next take the derivative of both sides of this equation with respect to p to get

$$\frac{1}{2} = \frac{1}{\sqrt{2\pi}}e^{-\varphi^2/2}\frac{d\varphi}{dp} \tag{3-63}$$

and, finally,

$$\frac{d\varphi}{dp} = \sqrt{\frac{\pi}{2}}e^{\varphi^2/2}. \tag{3-64}$$

Substituting equation (3-64) in equation (3-61) yields

$$\frac{\sigma^2(u_B)}{u_B^{\,2}} = \frac{u_L^{\,2}}{L^2} + \frac{1}{\varphi^2}\frac{\pi}{2}e^{\varphi^2}u_p^{\,2} \tag{3-65}$$

In applying equation (3-65), we are immediately confronted with the problem of obtaining u_L and u_p. These quantities can be estimated using any of the four formats described below.

Format 1: % of Values. This format reads "Approximately $C\%$ ($\pm\Delta c\%$) of observed values have been found to lie within the limits $\pm L$ ($\pm\Delta L$)."

In this format, a technical expert is asked to provide $\pm$ error for both the containment limits ($\pm\Delta L$) and the containment probability ($\pm \Delta c\%$). These limits are used to estimate u_A and u_p. The containment probability is

$$p = C / 100$$

where C is the percentage of values of y observed within $\pm L$.

If we assume that the errors in the estimates of L and p are approximately uniformly distributed within $\pm \Delta L$ and $\pm \Delta p = \pm \Delta c\% / 100$, respectively, then we can write

$$u_L{}^2 = \frac{(\Delta L)^2}{3} \qquad (3\text{-}66)$$

$$u_p{}^2 = \frac{(\Delta p)^2}{3} \qquad (3\text{-}67)$$

Use of the uniform distribution is appropriate here, since the ranges $\pm \Delta L$ and $\pm \Delta p$ can be considered analogous to "limits of resolution," for which the uniform distribution is applicable. This obviates the need for estimating confidence levels for ΔL and Δp. Any lack of rigor introduced by this tactic is felt as a third order effect and does not materially compromise the rigor of our final result.

Substituting equations (3-66) and (3-67) in equation (3-65) gives

$$\frac{\sigma^2(u_B)}{u_B{}^2} = \frac{(\Delta L)^2}{3L^2} + \frac{1}{\varphi^2}\frac{\pi}{2}e^{\varphi^2}\frac{(\Delta p)^2}{3} . \qquad (3\text{-}68)$$

Using equation (3-68) in equation (3-54) yields an estimate for the degrees of freedom, v_B, for a Type B uncertainty estimate.

$$v_B \cong \frac{1}{2}\left(\frac{\sigma^2(u_B)}{u_B{}^2}\right)^{-1}$$
$$\cong \frac{3\varphi^2 L^2}{2\varphi^2(\Delta L)^2 + \pi L^2 e^{\varphi^2}(\Delta p)^2} \qquad (3\text{-}69)$$

If ΔL and Δp are set to zero, then the Type B degrees of freedom becomes infinite. Obviously, in most cases, it is not realistic to have infinite degrees of freedom for Type B uncertainty estimates. Therefore, it behooves us to attempt to apply whatever means we have at our disposal to obtain a sensible estimate for p.

Format 2: X out of N. This format reads "Approximately X out of N observed values have been found to lie within the limits $\pm L$ ($\pm \Delta L$)."

In this format, the containment probability is expressed as $p = X / N$, where N is the number of observations of a value and X is the number of values observed to fall within $\pm L$ ($\pm \Delta L$). The variance in L is obtained the same as in Format 1. The variance in the containment probability p can be obtained by taking advantage of the binomial character of p.

$$u_p{}^2 = \frac{p(1-p)}{N} \qquad (3\text{-}70)$$

Substituting in equations (3-66) and (3-70) into equation (3-65) gives

$$\frac{\sigma^2(u_B)}{u_B^{\ 2}} = \frac{(\Delta L)^2}{3L^2} + \frac{1}{\varphi^2}\frac{\pi}{2}e^{\varphi^2}\frac{p(1-p)}{N}. \tag{3-71}$$

Using equation (3-71) in equation (3-54) yields an estimate for the degrees of freedom, v_B, for a Type B uncertainty estimate.

$$v_B \cong \frac{1}{2}\frac{u_B^{\ 2}}{\sigma^2(u_B)}$$

$$\cong \frac{3\varphi^2 L^2}{2\varphi^2(\Delta L)^2 + \pi L^2 e^{\varphi^2}p(1-p)/N} \tag{3-72}$$

Format 3: % of Cases. This format reads "Approximately C% of N observed values have been found to lie within the limits ±L (±ΔL)."

This format is a variation of Format 2 in which the containment probability is stated in terms of a percentage C of the number of observations N, with $p = C / 100$. The equation for estimating the degrees of freedom is the same as for Format 2:

$$v_B \cong \frac{3\varphi^2 L^2}{2\varphi^2(\Delta L)^2 + \pi L^2 e^{\varphi^2}p(1-p)/N} \tag{3-73}$$

Format 4: % Range. This format reads "Between C_1% and C_2% of observed values have been found to lie between the limits ±L (±ΔL)."

This format is a variation of Format 1 in which a range of values is given for the containment probability, $p = C/100$, where $C = (C_1 + C_2)$ and ±$\Delta c = (C_2 - C_1)/2$. The equation for estimating the degrees of freedom is the same as for Format 1:

$$v_B \cong \frac{3\varphi^2 L^2}{2\varphi^2(\Delta L)^2 + \pi L^2 e^{\varphi^2}(\Delta p)^2}. \tag{3-74}$$

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CHAPTER 4

UNCERTAINTY COMBINATION

4.0 General

This chapter discusses how uncertainties are combined and correlations between error sources are accounted for in an uncertainty analysis.

4.1 Definitions

4.1.1 <u>Combined Uncertainty</u>. The uncertainty in the total error of a value of interest.

4.1.2 <u>Correlation Analysis</u>. An analysis that determines the extent to which two error sources influence one another. Typically the analysis is based on ordered pairs of values of the two error source variables.

4.1.3 <u>Covariance</u>. The expected value of the product of the deviations of two random variables from their respective means. The covariance of two *independent* variables is zero.

4.1.4 <u>Effective Degrees of Freedom</u>. The degrees of freedom for combined uncertainties computed from the Welch-Satterthwaite formula.

4.1.5 <u>Error Source</u>. A parameter, variable or constant that can contribute error to the determination of the value of a subject parameter. Examples include: measuring parameter bias, random error, resolution error, operator bias, computation error and environmental factors error.

4.1.6 <u>Error Source Correlation</u>. See Correlation Analysis

4.1.7 <u>Error Source Uncertainty</u>. The uncertainty in the error of a given source.

4.1.8 <u>Pareto Chart</u>. See Pareto Diagram.

4.1.9 <u>Pareto Diagram</u>. A bar chart that ranks the relative contribution of component uncertainties to the total combined uncertainty.

4.1.10 <u>Total Uncertainty</u>. See Combined Uncertainty.

4.1.11 <u>Variance</u>. (1) Population: The expectation value for the square of the difference between the value of a variable and the population mean. (2) Sample: A measure of the spread of a sample equal to the sum of the squared observed deviations from the sample mean divided by the degrees of freedom for the sample. Also referred to as the mean square error.

4.2 Variance Addition

UncertaintyAnalyzer incorporates the variance addition rule, discussed in Chapter 1, because it provides a logical approach for combining Type A and Type B uncertainties that accounts for correlations between error sources. The general form of the variance addition rule for the direct measurement of a quantity x that involves n error sources

$$x = x_{true} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_n$$

is expressed as

$$
\begin{aligned}
\mathrm{var}(\sum_{i=1}^{n} \varepsilon_i) &= \sum_{i=1}^{n} \mathrm{var}(\varepsilon_i) + \sum_{i=1}^{n}\sum_{j>i}^{n} \rho_{ij} u_i u_j \\
&= \sum_{i=1}^{n} u_i^{2} + \sum_{i=1}^{n}\sum_{j>i}^{n} \rho_{ij} u_i u_j
\end{aligned}
\tag{4-1}
$$

where u_i is the uncertainty in error source ε_i and ρ_{ij} is the correlation coefficient for error sources ε_i and ε_j.

4.3 Error Source Correlation

UncertaintyAnalyzer's Correlation Analysis Screen provides a tool for correlating error sources for direct measurement, multivariate measurement and system model analyses. Correlation coefficients can be entered for each selected pair of error sources.

Alternatively, the correlation coefficient can be computed from sample data pairs entered in the Correlation Data table. The correlation coefficient for pairs of errors, ε_x and ε_y, is computed from

$$
\rho_{\varepsilon_x, \varepsilon_y} = \frac{\mathrm{cov}(\varepsilon_x, \varepsilon_y)}{u_x, u_y}
\tag{4-2}
$$

where the covariance of ε_x and ε_y is

$$
\mathrm{cov}(\varepsilon_x, \varepsilon_y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}) \sum_{i=1}^{n} (y_i - \bar{y}).
\tag{4-3}
$$

The Correlation Analysis Screen includes a Compensating Biases box that can be checked if the bias or error of one measured variable offsets the bias or error of another measured variable. For instance, if the same measuring parameter is used to measure the inside diameter of a sleeve and the outside diameter of a shaft that fits into the sleeve, any error or bias in the two measurements will not affect the quality of fit. In other words, the measurement biases offset each other.

4.4 Total Uncertainty and Degrees of Freedom

The total or combined uncertainty, u_T, is computed by taking the square root of the variance.

$$
u_T = \sqrt{\sum_{i=1}^{n} u_i^{2} + 2\sum_{i=1}^{n}\sum_{j>i}^{n} \rho_{ij} u_i u_j}
\tag{4-4}
$$

The effective degrees of freedom, v_{eff}, for the total uncertainty, u_T, resulting from the combination of uncertainties u_i with associated degrees of freedom, v_i, for n error sources is estimated using the Welch-Satterthwaite formula.

$$
v_{eff} = \frac{u_T^{4}{}_8}{\displaystyle\sum_{i}^{n} \frac{u_i^{4}}{v_i}}
\tag{4-5}
$$

The combined uncertainty u_{T*} is computed assuming no error source correlations. Consequently, the effective degrees of freedom is considered an estimate.

4.5 Pareto Diagrams

UncertaintyAnalyzer's Pareto Chart screen displays a bar chart that ranks the relative contribution of component uncertainties to the total combined measurement uncertainty. Each error source uncertainty is weighted in accordance with its magnitude and the extent that it is correlated with other error source uncertainties.

The Pareto Chart screen is accessed from screens and worksheets that account for multiple error sources. Some examples include digital sampling error, environmental error, stress response error, and computation error.

Because of the fact that certain error sources may be correlated with others, the percent contribution of an error source uncertainty to the total combined uncertainty cannot be obtained by simply dividing the error source uncertainty by the total uncertainty and multiplying by 100. Instead, the percent contribution is obtained by extracting each uncertainty in turn and computing the impact on the total.

4.6 Confidence Limits

In UncertaintyAnalyzer, the terms confidence limits and expanded uncertainty are used interchangeably. Both are computed as upper and lower limits that contain the true value μ (estimated) and expressed as

$$\bar{x} - t_{\alpha/s,v} u \leq \mu \leq \bar{x} + t_{\alpha/s,v} u \tag{4-6}$$

in which the standard uncertainty, u, is multiplied by a t-statistic, $t_{\alpha/2,v}$, that is computed for a given degrees of freedom, v, and probability or confidence level, p, where $\alpha = 1 - p$.

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CHAPTER 5

MULTIVARIATE UNCERTAINTY ANALYSIS

5. 0 General

This chapter discusses the approach used to estimate the uncertainty of a quantity (or subject parameter) that is computed from measurements of two or more attributes or parameters. The multivariate uncertainty analysis procedure consists of the following steps:

4. Develop the Parameter Value Equation

5. Develop the Error Model

6. Develop the Uncertainty Model

7. Identify the Measurement Process Errors

8. Estimate Measurement Process Uncertainties

9. Account for Error Source Correlations

10. Combine Uncertainties

The processes for developing error models and uncertainty models from the parameter value equation are presented. Identifying measurement process errors, estimating their uncertainties and accounting for cross-correlations are also presented. The volume of a cylinder obtained from length and diameter measurements is used to illustrate the concepts and methods of conducting a multivariate uncertainty analysis.

5. 1 Definitions

5.1.1 <u>Coefficient Equation</u>. An equation that expresses the partial derivative of a parameter value equation or module output equation with respect to a selected parameter or error source. This equation is used to compute the sensitivity coefficient for the selected parameter or error source.

5.1.2 <u>Combined Uncertainty</u>. The uncertainty in the total error of a value of interest.

5.1.3 <u>Component Error</u>. The error in the measurement of a given component of a multivariate measurement. For example, when expressing cylinder volume as a function of length and diameter components, the error in the cylinder volume measurement, ε_V, can be expressed in terms of the component errors, ε_L and ε_D.

5.1.4 <u>Component Uncertainty</u>. The product of the sensitivity coefficient and the standard uncertainty for a component error.

5.1.5 <u>Computed Parameter Value</u>. The parameter value computed on UncertaintyAnalyzer's Multivariate Analysis Screen. Based on user specified adjusted mean values for root variables and the Parameter Value Equation.

5.1.6 <u>Cross-correlation</u>. The correlation between two error sources for two different components of a multivariate analysis. For example, if the same person (operator) makes measurements of both component X and component Y, then there is a cross-correlation between the operator biases for two components.

5.1.7 <u>Compensating Biases</u>. Measuring parameter biases that offset or compensate one another. For example, if the same measuring parameter is used to measure the inside diameter of a sleeve and the outside diameter of a shaft that fits into the sleeve, any error or bias in the two measurements will not affect the quality of fit.

5.1.8 <u>Effective Degrees of Freedom</u>. The degrees of freedom for combined uncertainties computed from the Welch-Satterthwaite formula.

5.1.9 <u>Multivariate Measurements</u>. Measurements in which the subject parameter is a computed quantity based on measurements of two or more attributes or parameters.

5.1.10 <u>Nested Variables</u>. Variables that are defined as a function of root variables or other nested variables.

5.1.11 <u>Nested Variables Equations</u>. Equations that define variables in terms of root variables or other nested variables. They are entered after the parameter value equation.

5.1.12 <u>Parameter Value Equation</u>. A mathematical expression that defines the value of a measurement or parameter value in terms of the values of constituent root variables or error sources.

5.1.13 <u>Root Variables</u>. Variables used to compute the parameter value that are not a function of other variables. Root variable information is specified by the user via the Error Source Worksheets.

5.1.14 <u>Sensitivity Coefficient</u>. For a multivariate analysis, the sensitivity coefficient of a given root variable is the partial derivative of the parameter value equation with respect to a root variable. For a system analysis, the sensitivity coefficient for a given module parameter is the partial derivative of the module output equation with respect to a module parameter.

5.1.15 <u>System Equation</u>. A mathematical expression that defines the value of a quantity in terms of its constituent variables or components.

5.1.16 <u>Total Uncertainty</u>. See Combined Uncertainty.

5.2 Parameter Value Equation

The parameter value equation is a mathematical relationship between the quantity of interest (subject parameter) and the variables or components to be measured. The parameter value equation is also referred to as the system or governing equation.

For example, suppose we wish to know the volume of a cylinder. The parameter value equation for the cylinder volume is given as

$$V = \pi L \left(\frac{D}{2} \right)^2 \tag{5-1}$$

where L and D are the cylinder length and diameter, respectively. From this equation, we see that, to determine the cylinder volume, we need to measure length and diameter components.

In UncertaintyAnalyzer, the parameter value is computed based on user specified mean or nominal values for root variables included in the parameter value equation and other nested variables equations.

5.3 Error Model

In any given measurement scenario, each measured quantity is a potential source of error. For example, errors in the length and diameter measurements will contribute to the overall error in the estimation of the cylinder volume. Therefore, the cylinder volume equation can be expressed as

$$V_0 + \varepsilon_V = \pi \left(\frac{D_0 + \varepsilon_D}{2} \right)^2 (L_0 + \varepsilon_L) \tag{5-2}$$

where

V_0 = true or nominal cylinder volume

D_0 = true or nominal cylinder diameter

L_0 = true or nominal cylinder length

ε_V = error in the cylinder volume

ε_D = error in the cylinder diameter

ε_L = error in the cylinder length

By rearranging equation (5-2), we obtain an algebraic expression for the cylinder volume error.

$$\begin{aligned}
\varepsilon_V &= \pi \left(\frac{D_0 + \varepsilon_D}{2} \right)(L_0 + \varepsilon_L) - V_0 \\
&= \pi \left(\frac{D_0 + \varepsilon_D}{2} \right)(L_0 + \varepsilon_L) - \pi \left(\frac{D_0}{2} \right)^2 L_0 \\
&= \frac{\pi}{4} \left(D_0^2 + 2D_o \varepsilon_D^2 \right)(L_0 + \varepsilon_L) - \pi \left(\frac{D_0}{2} \right)^2 L_0 \\
&= \frac{\pi}{4} \left(L_0 D_0^2 + 2L_0 D_0 \varepsilon_D + L_0 \varepsilon_D^2 + D_0^2 \varepsilon_L + 2D_0 \varepsilon_D \varepsilon_L + \varepsilon_D^2 \varepsilon_L \right) - \pi \left(\frac{D_0}{2} \right)^2 L_0
\end{aligned} \tag{5-3}$$

The terms $L_0 \varepsilon_D^2$, $2D_0 \varepsilon_D \varepsilon_L$, and $\varepsilon_D^2 \varepsilon_L$ are referred to as second order terms and are considered to be small compared to the other first order terms. Neglecting these terms, the cylinder volume error equation can be expressed in a simpler form.

$$\varepsilon_V = \frac{\pi}{4} \left(L_0 D_0^2 + 2L_0 D_0 \varepsilon_D + D_0^2 \varepsilon_L \right) - \pi \left(\frac{D_0}{2} \right)^2 L_0 \tag{5-4}$$

Rearranging equation (5-4), we can further simplify the equation for ε_V.

$$\varepsilon_V = \pi\left(\frac{D_0}{2}\right)^2 \varepsilon_L + \pi L_0 \frac{D_0}{2} \varepsilon_D + \pi\left(\frac{D_0}{2}\right)^2 L_0 - \pi\left(\frac{D_0}{2}\right)^2 L_0$$

$$= \pi\left(\frac{D_0}{2}\right)^2 \varepsilon_L + \pi L_0 \frac{D_0}{2} \varepsilon_D \tag{5-5}$$

The coefficients for ε_L and ε_D in equation (5-5) are actually the partial derivatives of V with respect to L and D.

$$\frac{\partial V}{\partial L} = \pi\left(\frac{D}{2}\right)^2 \quad and \quad \frac{\partial V}{\partial D} = \pi L\left(\frac{D}{2}\right)$$

Therefore, the cylinder volume error model can be expressed as

$$\varepsilon_V = \left(\frac{\partial V}{\partial L}\right)\varepsilon_L + \left(\frac{\partial V}{\partial D}\right)\varepsilon_D \tag{5-6}$$

where the partial derivatives are sensitivity coefficients that determine the relative contribution of the errors in length and diameter to the total error.

5.4 Uncertainty Model

Axiom 3 provides the direct link between error and uncertainty that we need to quantify measurement uncertainty. It is restated here for convenience.

Axiom 3: The uncertainty in a quantity or variable is the square root of the variable's mean square error or variance. In mathematical terms, this is expressed as

$$u_V = \sqrt{\text{var}(\varepsilon_V)} \tag{5-7}$$

And, according to the **variance addition rule**, the variance in ε_V can be expressed in terms of the variances in ε_L and ε_D.

$$u_V = \sqrt{\text{var}(\varepsilon_V)}$$

$$= \sqrt{\text{var}(c_L \varepsilon_L + c_D \varepsilon_D)} \tag{5-8}$$

$$= \sqrt{c_L^2 u_L^2 + c_D^2 u_D^2 + \rho_{LD} u_L u_D}$$

where ρ_{LD} is the correlation coefficient for the uncertainties in the length and diameter component errors and c_L and c_D are the sensitivity coefficients.

$$c_L = \left(\frac{\partial V}{\partial L}\right) \quad and \quad c_D = \left(\frac{\partial V}{\partial D}\right) \tag{5-9}$$

UncertaintyAnalyzer automatically computes partial derivatives once the user has defined the parameter value equation in mathematical terms. Alternatively, the user has the option of entering a coefficient equation for a component or variable. The protocol for entering an equation in the Sensitivity Coefficient Equation field is the same as for the Parameter Value

Equation field.

5.4.1 Sensitivity Coefficients. The following discussion illustrates the method used within UncertaintyAnalyzer to compute partial derivatives of a function $f(x, y, z, \cdots)$ with respect to a variable x

$$\frac{\partial f(x, y, z, \cdots)}{\partial x} \equiv \underset{\delta x \to 0}{Lim} \frac{f(x + \delta x, y, z, \cdots) - f(x, y, z, \cdots)}{\delta x}, \qquad (5\text{-}10)$$

where the *Lim* symbol reads "in the limit as δx goes to zero."

As an example, consider the function

$$f(x, y) = xy.$$

Using the above definition, we have

$$\frac{\partial f(x, y)}{\partial x} \equiv \underset{\delta x \to 0}{Lim} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

$$= \underset{\delta x \to 0}{Lim} \frac{(x + \delta x)y - xy}{\delta x}$$

$$= \underset{\delta x \to 0}{Lim} \frac{xy + (\delta x)y - xy}{\delta x}$$

$$= \underset{\delta x \to 0}{Lim} \frac{(\delta x)y}{\delta x} = y.$$

In this case, the partial derivative does not contain the variable x. However, it is easy to see that the sensitivity coefficient for x is the variable y. That is, the larger or smaller y is the more or less the impact of changes in x.

As another example, consider the function

$$f(x, y) = x^2 y.$$

In this case, the partial derivative with respect to x is

$$\frac{\partial f(x, y)}{\partial x} \equiv \underset{\delta x \to 0}{Lim} \frac{(x + \delta x)^2 y - x^2 y}{\delta x}$$

$$= \underset{\delta x \to 0}{Lim} \frac{x^2 y + 2x(\delta x)y + (\delta x)^2 - x^2 y}{\delta x}$$

$$= \underset{\delta x \to 0}{Lim} (2xy + \delta x)$$

$$= 2xy.$$

In this instance, the variable x is included in the partial derivative. But how about the variable y? For this, we have

$$\frac{\partial f(x,y)}{\partial y} \equiv \lim_{\delta y \to 0} \frac{x^2(y+\delta y) - x^2 y}{\delta y}$$
$$= \lim_{\delta y \to 0} \frac{x^2 y + x^2(\delta y) - x^2 y}{\delta y}$$
$$= x^2,$$

which is function of x only.

5.5 Measurement Process Errors

The errors in the length and diameter components, ε_L and ε_D, can be expressed in terms of their constituent process errors.

$$\varepsilon_L = \varepsilon_{Lbias} + \varepsilon_{Lran} + \varepsilon_{Lres} + \varepsilon_{Lop} + \varepsilon_{Lenv} \tag{5-11}$$

$$\varepsilon_D = \varepsilon_{Dbias} + \varepsilon_{Dran} + \varepsilon_{Dres} + \varepsilon_{Dop} + \varepsilon_{Denv} \tag{5-12}$$

The process error sources are:

Bias (bias)	-	the bias in the micrometer readings.
Random (ran)	-	the error that produces different results from measurement to measurement.
Resolution (res)	-	the error due to the finite resolution of the micrometer readings.
Operator (op)	-	the error due to any systematic bias on the part of the measuring technician.
Environment (env)	-	the error in any thermal or other correction due to a departure from nominal conditions.

Measurement process errors are the basic elements of uncertainty analysis. Once these fundamental error sources have been identified, we can then begin to develop uncertainty estimates.

5.6 Measurement Process Uncertainties

The uncertainty in each component is expressed in terms of the uncertainties in the error sources obtained using Axiom 3 and the variance addition rule. Operating on ε_L above with the variance operator gives, for the uncertainty in the length measurement,

$$u_L = \sqrt{u_{Lbias}^2 + u_{Lran}^2 + u_{Lres}^2 + u_{Lop}^2 + u_{Lenv}^2} \tag{5-13}$$

Likewise, applying the variance operator to ε_D above gives, for the uncertainty in the diameter measurement,

$$u_D = \sqrt{u_{Dbias}^2 + u_{Dran}^2 + u_{Dres}^2 + u_{Dop}^2 + u_{Denv}^2} \tag{5-14}$$

Notice that there are no terms correlating process uncertainties within each component expression. This is because the length measurement process errors are independent of one another, as are the diameter measurement process errors.

However, some of the length measurement process errors may not be independent of some of the diameter measurement process errors. Therefore, we must consider possible cross-correlation terms. This will be addressed in the next section.

The methods of uncertainty estimation are

Bias (bias)	- heuristically from tolerance limits and in-tolerance probabilities.
Random (ran)	- statistically from a measurement sample.
Resolution (res)	- heuristically from the measuring parameter resolution spec and assumptions about containment probability.
Operator (op)	- heuristically as a function of measuring parameter resolution.
Environment (env)	- heuristically from tolerances and in-tolerance probabilities for the environment monitoring equipment.

In UncertaintyAnalyzer, measurement process uncertainties are estimated using the Error Source Worksheets accessed from the Multivariate Analysis Screen. The Error Source Worksheets provide a useful tool for making both Type A and Type B uncertainty estimates.

5.7 Error Source Correlations

Before we combine uncertainties, we must consider if there are any possible cross-correlations between process uncertainties for the two components. First, let us review what we know about the cylinder measurement process.

1. Both length and diameter are measured using the same device (i.e., a micrometer).

2. All measurements are made by the same person (operator).

3. All measurements were made in the same measuring environment.

Given this knowledge, we can assert that the following process uncertainties are cross-correlated between the length and diameter components:

▪ Measurement Bias - u_{Lbias} and u_{Dbias}

▪ Operator Bias - u_{Lop} and u_{Dop}

▪ Environmental Factors - u_{Lenv} and u_{Denv}

Second, we need to write the an equation that expresses the correlation coefficient, ρ_{LD}, for the component uncertainties, u_L and u_D, in terms of the correlation coefficients for the process uncertainties that are cross-correlated between components. The expression is

$$\rho_{xy} = \frac{1}{u_x u_y} \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} \rho\left(\varepsilon_{xi}, \varepsilon_{xj}\right) u_{xi} u_{xj} \tag{5-15}$$

where u_L and u_D are the total component uncertainties and u_{Li} and u_{Dj} are the process uncertainties for the length and diameter components, respectively. (subscripts don't match equation 5-15)

Correlation coefficients range from minus one to plus one. A positive correlation coefficient applies when the error sources are directly related. A negative correlation coefficient is used when the error sources are inversely related.

UncertaintyAnalyzer's Correlation Analysis Screen allows you to compute correlation coefficients from pairs of observations. Alternatively, if you have a pretty good estimate of a correlation coefficient, you can enter it directly.

For the cylinder volume, we need only consider the cross-correlations between component measurement bias uncertainties, operator bias uncertainties, and environmental factors uncertainties. Let us consider the values for these three correlation coefficients.

5.7.1 Correlation between Component Measurement Biases. Since the same device is used to measure both length and diameter, the parameter bias for these measurements is the same. In this instance, the correlation coefficient, $\rho_{Lbias,Dbias}$, is equal to 1.0

5.7.2 Correlation between Component Operator Biases. Although the same operator makes both measurements, human inconsistency prevents us from assigning a correlation coefficient equal to 1.0. However, we also know that the correlation coefficient should not be equal to zero either. Given that this is all we can say from heuristic considerations, we will set the correlation coefficient between length and diameter operator biases, $\rho_{Lop,Dop}$, equal to 0.5

5.7.3 Correlation between Component Environmental Factors Errors. Since the length and diameter measurements are made in the same environment, the correlation coefficient between the length and diameter environmental factors, $\rho_{Lenv,Denv}$, is also equal to 1.0.

5.8 Total Uncertainty and Degrees of Freedom

We can now expand the total uncertainty equation for cylinder volume, u_V, in terms of the process uncertainties.

$$
\begin{aligned}
u_V^2 &= c_L^2 u_L^2 + c_D^2 u_D^2 + 2\rho_{LD} u_L u_D \\
&= c_L^2 \left(u_{Lbias}^2 + u_{Lran}^2 + u_{Lres}^2 + u_{Lop}^2 + u_{Lenv}^2 \right) + c_D^2 \left(u_{Dbias}^2 + u_{Dran}^2 + u_{Dres}^2 + u_{Dop}^2 + u_{Denv}^2 \right) \\
&\quad + 2 c_L c_D \left(\rho_{Lbias,Dbias} u_{Lbias} u_{Dbias} + \rho_{Lop,Dop} u_{Lop} u_{Dop} + \rho_{Lenv,Denv} u_{Lenv} u_{Denv} \right)
\end{aligned}
$$

UncertaintyAnalyzer automatically combines the uncertainties using the appropriate sensitivity coefficients and accounting for correlations between error source uncertainties.

When uncertainties are combined, we need to know the degrees of freedom for the total uncertainty. We can compute the degrees of freedom for the uncertainty in the cylinder volume, u_V, from the Welch-Satterthwaite formula

$$
\nu = \frac{u_{V*}^4}{\dfrac{c_L^4 u_L^4}{\nu_L} + \dfrac{c_D^4 u_D^4}{\nu_D}}
\tag{5-16}
$$

where u_{V*} is the total uncertainty computed without cross-correlations between component process uncertainties.

$$
u_{V*} = \sqrt{c_L^2 u_L^2 + c_D^2 u_D^2}
\tag{5-17}
$$

The degrees of freedom for the component uncertainties are

$$v_L = \frac{u_L^4}{\dfrac{u_{Lbias}^4}{v_{Lbias}} + \dfrac{u_{Lran}^4}{v_{Lran}} + \dfrac{u_{Lres}^4}{v_{Lres}} + \dfrac{u_{Lop}^4}{v_{Lop}} + \dfrac{u_{Lenv}^4}{v_{Lenv}}} \qquad (5\text{-}18)$$

and

$$v_D = \frac{u_D^4}{\dfrac{u_{Dbias}^4}{v_{Dbias}} + \dfrac{u_{Dran}^4}{v_{Dran}} + \dfrac{u_{Dres}^4}{v_{Dres}} + \dfrac{u_{Dop}^4}{v_{Dop}} + \dfrac{u_{Denv}^4}{v_{Denv}}} \qquad (5\text{-}19)$$

In UncertaintyAnalyzer, the effective degrees of freedom for the total uncertainty is automatically computed using the Welch-Satterthwaite formula.

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CHAPTER 6

SYSTEM UNCERTAINTY ANALYSIS

6.0 General

This chapter discusses the approach used to estimate the uncertainty of a quantity (or subject parameter) that is measured with a system comprised of component modules arranged in series. The analysis process traces system uncertainty module by module from system input to system output.
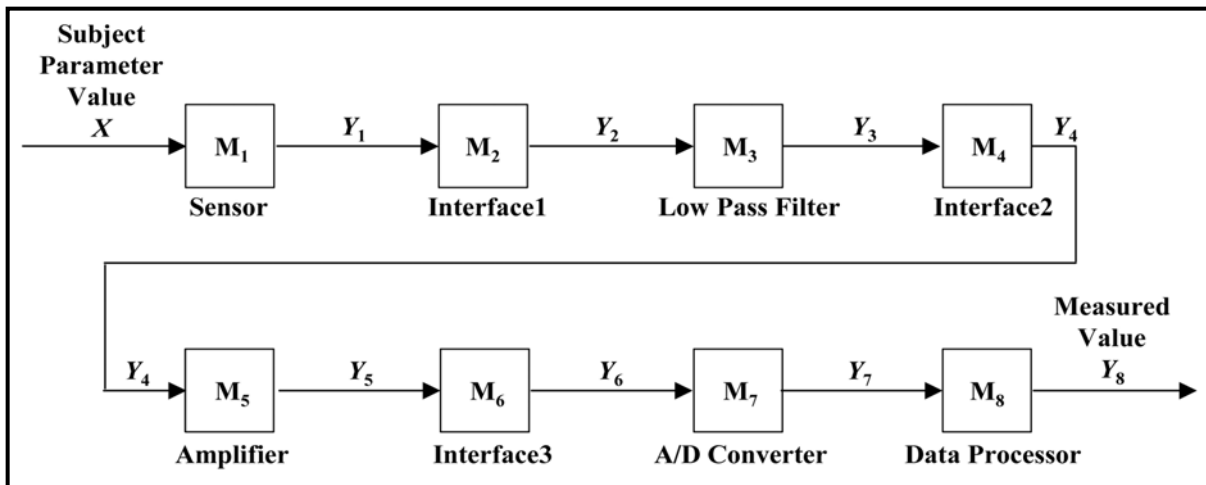
Figure 6-1. Block Diagram for Example System.

System uncertainty analysis follows a structured procedure. This is necessary because the output from any given module of a system may comprise the input to another module or modules. Since each module's output carries with it an element of uncertainty, this means that the same uncertainty may be present at the input of some other module.

In analyzing linear system models, we develop output equations for each module. From these equations, we identify sources of error for each module. We then estimate the uncertainty in each error source and the combined uncertainty in the output of each module. In doing this, we make certain that the uncertainty in the output of each module was included in the input to the succeeding module in the system.

In this respect, the system analysis results are computed somewhat differently that those previously discussed for direct measurements and multivariate measurements. The general system analysis procedure consists of the following steps:

1. Develop a Measurement System Model — Determine the measurement system stages or modules involved in processing the measurement of interest. Identify the hardware and software used.

2. Develop Module Equations — Develop the set of equations that describe module outputs in terms of inputs and identify the parameters that characterize these processes.

3. Identify Module Error Sources — From the module output equations, identify and describe functions or parameters that may contribute to the error in the module output value.

4. Estimate Module Uncertainties — Compute output values, uncertainties and associated degrees of freedom for each module, accounting for correlations between error sources.

5. Estimate Total System Uncertainty - Propagate module output values and uncertainties to determine the system output, uncertainty and associated degrees of freedom.

The processes for developing a system model and the corresponding module output equations are presented. Procedures for identifying measurement process errors, estimating their uncertainties and accounting for correlations are also presented using a temperature measurement system for illustration.

6.1 Definitions

6.1.1 Module Error Sources. Sources of error that accompany the conversion of module input to module output.

6.1.2 Module Input Uncertainty. The uncertainty in a module's input expressed as the uncertainty or standard deviation in the output of the preceding module.

6.1.3 Module Output Equation. The equation that expresses the output from a module in terms of its input. The equation is characterized by parameters that represent the physical processes that participate in the conversion of module input to module output.

6.1.4 Module Output Uncertainty. The total combined uncertainty in the output of a given module of a measurement system.

6.1.5 System Module. A module or stage of a system involved in processing a measurement.

6.1.6 System Output Uncertainty. The total uncertainty in the output of a measurement system.

6.1.7 Total Module Uncertainty. See Module Output Uncertainty.

6.1.8 Total System Uncertainty. See System Output Uncertainty.

6.2 System Model

The first step in the system analysis procedure is to develop a model that describes the modules involved in processing the measurement of interest (i.e., subject parameter). The model should include a diagram depicting the modules of the system and their inputs and outputs and identify the hardware and software used. The system diagram can be a useful guide for developing the equations that describe the module outputs in terms of inputs and identify the parameters that characterize these processes. It may also be beneficial to develop a functional model that relates component errors to the overall system error.

For illustrative purposes, we will evaluate a temperature measurement system for converting a time-varying analog value to a digital representation. A number of specialized disciplines are involved, so we will go into some detail about the physical processes.[14]

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[14] A good discussion of measurement systems can also be found in *Computer-Based Data Acquisition Systems Design Techniques*, Taylor, J.L., Instrument Society of America, 1986.

The digital temperature measurement system is designed to pro vide the folloing capabilities:

Temperature Range: 20–25 ºC
Bandwidth: 0–10 Hz (sine wave)
Mode of Heat Transfer: Natural convection from fluid to probe,
                        conduction from probe to thermocouple.
Measurement Sensor: Type K Chromel-Alumel Thermocouple

The system diagram should be detailed enough to depict key components or modules, their inputs and outputs, and identify the hardware and software used. An example diagram for the digital temperature measurement system is shown below.
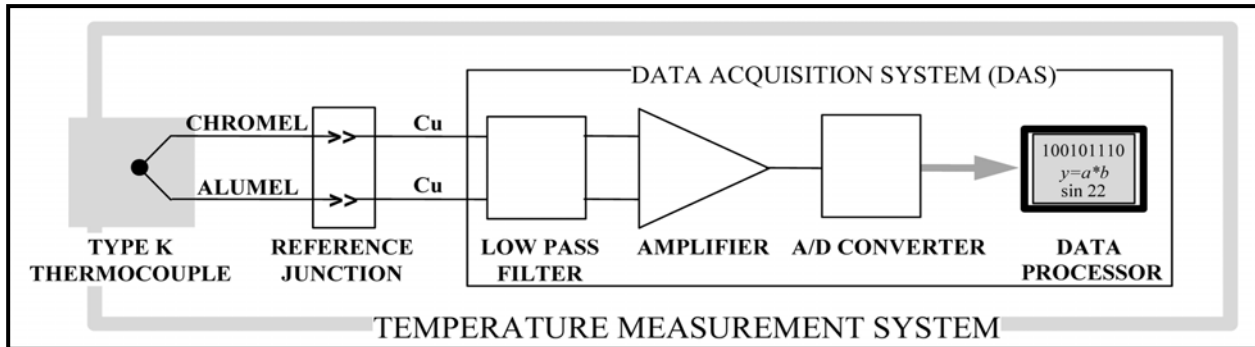
Figure 6-2. Temperature Measurement System Diagram.

A block diagram for the temperature measurement system should also be developed to identify and label each module, as well as the input and output for each module. Interface modules are included to account for any gain/losses as the signal passes from one system component to another.
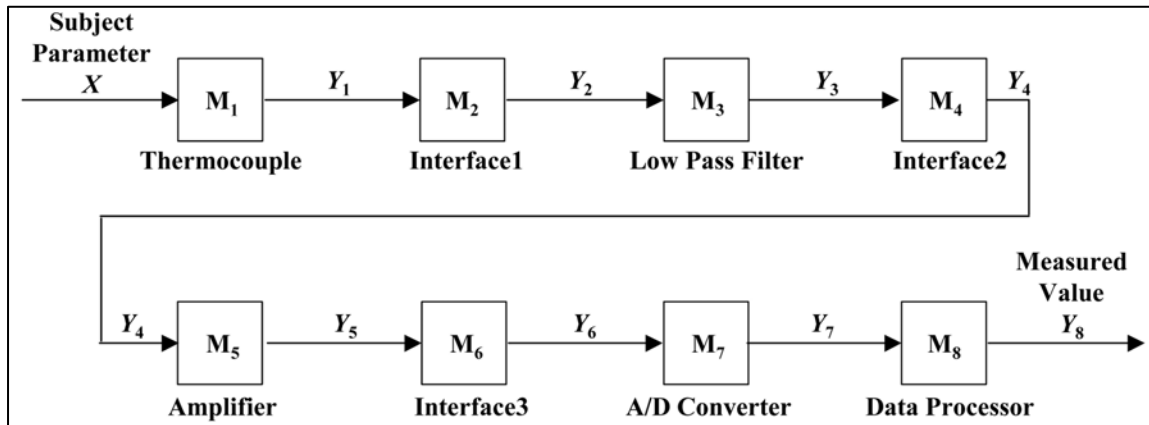
Figure 6-3. Block Diagram for Temperature Measurement System.

For instance, there may be errors introduced due to signal attenuation, noise, or cross-talk at the connection between the thermocouple Reference Junction output terminals and the input terminals of the Low Pass Filter. The Interface1 module has been inserted between the Thermocouple and Low Pass Filter modules to account for such errors. Similarly, the Interface2 and Interface3 modules have been added to account for errors introduced at the interfaces between the Low Pass Filter and Amplifier modules and between the Amplifier and the A/D Converter modules, respectively.

6.3 Module Output Equations

Once a sufficiently detailed block diagram has been established, we can develop the equations that relate the inputs and outputs for each module. The basic approach is to clearly describe the physical processes and identify sources of error that can affect the measured value along its path from module to module.

6.3.1   Thermocouple Sensor Module (M_1). In developing the equation to compute the output value, Y_1, as a function of the input X, we need to consider the relevant variables or parameters that need to be included. The thermocouple sensor module consists of the chromel-alumel measuring junction, a reference junction and output copper leads, as shown in Figure 6-4.
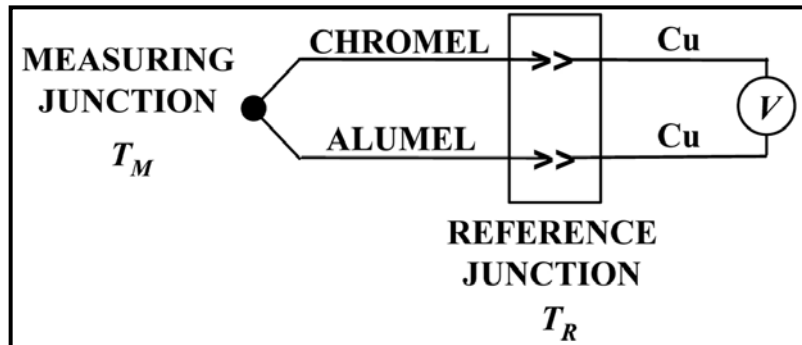
Figure 6-4. Schematic of Thermocouple Sensor Module (M1).

When two wires of different materials, such as chromel and alumel, are joined at one end and placed in a temperature gradient, a thermoelectric voltage is observed at the other end. This is the common operating principle of a thermocouple and is described by the Seebeck effect.

Ideally, if the wires are made of thermoelectrically homogenous material, the temperature of the measuring junction (T_M) can be obtained from the measured voltage (V), knowledge of the Seebeck coefficients of the wires, and knowledge about the temperature of the wires at the reference junction.

In reality, however, errors arise from numerous avoidable and unavoidable sources that are listed below.

- Electrical shunting
- Drift, aging, and hysteresis
- Noise
- Calibration or sensitivity errors
- Reference junction errors

6.3.1.1 <u>Electrical Shunting</u>. If the electrical resistance of the insulation between the thermo-couple wires degrades appreciably, then cross-conduction can result in the formation of virtual temperature junctions. The severity of electrical shunting depends upon the properties and thickness of the insulating material and the thickness of the thermocouple wires.

For the purposes of the temperature measurement system example, we will assume that the Type K thermocouple module does not experience any errors due to electrical shunting.

6.3.1.2 <u>Drift and Aging</u>. Thermocouples that are in service for extended time periods at elevated temperatures can change characteristics. The changes depend on the initial purity of the thermocouple material, contaminants introduced from the environment, and temperature.

No general estimate of the rate of thermocouple drift can be stated. However, we know that in some instances, thermocouples have been found to be stable for thousands of hours of continuous exposure to high temperature, while others under similar conditions drift appreciably within a few hours.

6.3.1.3 <u>Hysteresis</u>. Hysteresis is the variation in the thermocouple output voltage for a specific temperature input when temperature is approached from different directions, as illustrated below. When included as a performance specification, it is stated as the maximum difference in the output voltage for the same temperature during a full range traverse in each direction.
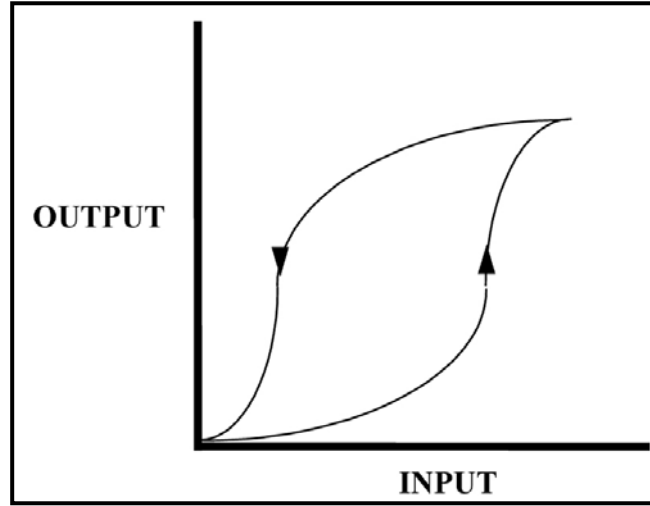
Figure 6-5. Hysteresis Effect.

Hysteresis error arises from residual values left over from previous samples. Represent this quantity by the variable V_h. If the sampling rate is v_s, then

$$V_h = V\left(1 - e^{-\lambda \tau}\right) e^{-\lambda(T-\tau)} \qquad (6\text{-}1)$$

where $T = 1/v_s$.

The hysteresis error is the sampled value of this residual amount. Thus the hysteresis error is given by

$$\varepsilon_h(t) = V_h\left(1 - e^{-\lambda t}\right) \qquad (6\text{-}2)$$

If the sample point t is unknown, we instead use the average error

$$\bar{\varepsilon}_h = V_h\left[1 - \frac{1}{\lambda \tau}\left(1 - e^{-\lambda \tau}\right)\right] \qquad (6\text{-}3)$$

The hysteresis uncertainty can be obtained from the hysteresis error in the same way as the impulse response uncertainty was obtained. If the sample point is known, then

$$\sigma_h = \lambda V_h e^{-\lambda t} \sigma_t \qquad (6\text{-}4)$$

whereas, if the sample point is unknown,

$$
\begin{aligned}
\sigma_h^2 &= \frac{1}{\tau}\int_0^\tau \varepsilon_h^2(t)\,dt - \bar{\varepsilon}_h^2 \\
&= V_h^2\left[1 - \frac{2}{\lambda \tau}\left(1 - e^{-\lambda \tau}\right) + \frac{1}{2\lambda \tau}\left(1 - e^{-\lambda \tau}\right)\right] - \bar{\varepsilon}_h^2 \qquad (6\text{-}5) \\
&= V_h^2\left[\frac{1}{2}\left(1 - e^{-\lambda \tau}\right) + \frac{1}{\lambda \tau}\left(1 - e^{-\lambda \tau}\right)^2\right].
\end{aligned}
$$

6.3.1.4 <u>Noise</u>. Since the thermocouple leads are conductors, externally applied electromagnetic fields may introduce stray emfs that contaminate the voltage output due to the temperature between the measuring junction and reference junction. Such "noise" comprises an error. Noise is usually random in character.

UncertaintyAnalyzer accounts for *thermal noise* and *shot noise* in electronic circuits. Provision is also made for user-estimated noise due to other effects. Note, that, since noise is, in itself error, the uncertainty due to a given source of noise is the square root of the variance of the noise distribution.

6.3.1.4.1 <u>Thermal Noise</u>. The mean square voltage due to noise produced by delivering current to a load with resistance R is given by

$$V_{thermal}^2 = 4k_B TR\Delta f \tag{6-6}$$

where

$$
\begin{aligned}
k_B &= \text{Boltzmann's constant} \\
T &= \text{temperature of the load} \\
R &= \text{resistance of the load} \\
\Delta_f &= \text{bandwidth of the signal}
\end{aligned}
$$

Consequently, the uncertainty due to thermal noise is

$$u_{thermal} = \sqrt{4k_B TR\Delta f} \tag{6-7}$$

6.3.1.4.2 <u>Shot Noise</u>. Shot noise is the fluctuation in the current of charge carriers passing through a surface at statistically independent times. It has a uniform spectral density V_{shot} given by

$$V_{shot}^2 = 2eI_0 n\Delta f \tag{6-8}$$

where

$$
\begin{aligned}
e &= \text{electron charge} \\
I_0 &= \text{current passed through one or more pn junctions} \\
n &= \text{number of junctions} \\
\Delta f &= \text{bandwidth of the signal}
\end{aligned}
$$

The uncertainty due to shot noise is given by

$$u_{shot} = \sqrt{2eI_0\Delta f} \tag{6-9}$$

6.3.1.4.3 <u>Total Electronic Noise</u>. The uncertainty due to electronic noise is computed as the root-sum-square of $u_{thermal}$ and u_{shot}

$$u_e = \sqrt{u_{thermal}^2 + u_{shot}^2} \tag{6-10}$$

6.3.1.4.4 <u>Signal to Noise Ratio</u>. The total voltage due to electronic noise is computed as

$$V_e = \sqrt{V_{thermal}^2 + V_{shot}^2}$$
$$= u_e$$

(6-11)

Letting V_{sig} represent the rms amplitude of the signal, the signal to noise ratio is computed as

$$R_{sn} = \frac{V_{sig}}{V_e}$$

6.3.1.4.5 <u>Other Noise</u>. Users can account for uncertainty due to noise from other sources. This consists of a Type B uncertainty analysis in which the standard uncertainty is computed from error limits $\pm L$ and a confidence level c. Provision is made for five noise sources.

6.3.1.4.5.1 <u>Normal Noise Distribution</u>. If the confidence level is less that 100%, UncertaintyAnalyzer assumes a normal distribution. The noise contribution of the ith Other Noise source is given by

$$u_i = \frac{L}{\Phi^{-1}\left(\dfrac{1 + c/100}{2}\right)}$$

(6-13)

where

L	=	error limit
c	=	confidence level (%)
Φ^{-1}	=	inverse normal distribution function

6.3.1.4.5.2 <u>Uniform Noise Distribution</u>. If the confidence level is 100%, UncertaintyAnalyzer assumes a uniform distribution. The noise contribution of the ith Other Noise source is given by

$$u_i = \frac{L}{\sqrt{3}}.$$

(6-14)

6.3.1.4.5.3 <u>Combined Other Noise</u>. Each Other Noise uncertainty is weighted with a user-supplied coefficient a_i, $i = 1, 2, ..., 5$. Accordingly, the total uncertainty due to Other Noise is computed according to

$$u_{other} = \sqrt{\sum_{i=1}^{5} a_i^2 u_i^2}.$$

(6-15)

6.3.1.4.5 <u>Total Noise Uncertainty</u>. The total uncertainty due to noise is computed in rss:

$$u_{noise} = \sqrt{u_e^2 + u_{other}^2}.$$

(6-16)

6.3.1.5 <u>Calibration or Sensitivity Errors</u>. The relationship of the temperature of the measuring junction (T_M) and the measured voltage is tabulated in thermocouple calibration tables for common thermocouple types. These calibration tables generally assume a reference temperature (T_R) equal to 0 °C. If an ice bath is used as the reference junction or the thermocouple has a means of compensating for a reference junction temperature other than 0 °C, then the tempera-

ture of the measuring junction for a given voltage output can be obtained directly from these tables.

However, it is important to note that these thermocouple calibration (or reference) tables provide the expected characteristics for common thermocouple types. Even if thermocouples are manufactured in conformance with specified standards, there is some difference between the actual thermocouple response characteristic (e.g., Seebeck Coefficient) and that given in the reference tables. This difference introduces an error sensitivity of the thermocouple.

Alternatively, if the thermocouple is individually calibrated, errors arise from the fitting a calibration curve to a fixed set of test points and from the accuracy of the comparison thermometer. For this analysis example, we will assume that the Type K thermocouple has been manufactured according to standard specifications and has not been individually calibrated.

6.3.1.6 <u>Reference Junction Error</u>. It is important to remember that the voltage output of a thermocouple depends on the magnitude of the temperature difference between the measuring junction and the reference junction. The larger the temperature difference, the larger the voltage output. If the reference junction temperature is allowed to warm up above 0 °C, while the measuring junction temperature is held at a given temperature, the voltage output of the thermocouple will decrease. For this analysis example, we will assume that an ice bath has been used to achieve a reference junction temperature of 0 °C.

Based on our above assessment of the thermocouple sensor module error sources and assuming that hysteresis and drift can be expressed as fractions of the input temperature, X, the output equation for the thermocouple module M_1 can be expressed as

$$Y_1 = (p_{11} + p_{13})(1 + p_{12})(X + p_{15}) + p_{14}$$

where

Table 6-1. Parameters Used in Thermocouple Module Output Equation			
Parameter Name	Parameter Description	Nominal Value	Error Limits
X	Input value (i.e., Subject Parameter)	22.0 °C	
p_{11}	Sensitivity (i.e., Seebeck Coefficient)	40.4055 µV/°C	±0.0925 µV/°C
p_{12}	Thermocouple hysteresis	0 µV	±0 µV
p_{13}	Thermocouple drift	0 µV	±0 µV
p_{14}	Noise	0 µV	±0 µV
p_{15}	Reference junction deviation from 0 °C	0 °C	±0.27 0 °C

The output equation for a selected module is entered into the Module Output Equation field of UncertaintyAnalyzer's System Model Screen. Module output equations are entered in VB Script format, which is a simplified form of VBA (Visual Basic for Applications). The module output equation can contain numbers, constants, variables, functions, and mathematical operators.

The system input, X, is the subject parameter of interest that we are measuring. In this case, we are measuring temperature. In UncertaintyAnalyzer, the subject parameter is defined by entering descriptive information and other specifications (as appropriate) into the Subject Parameter Bias Uncertainty Worksheet.

6.3.2 Interface1 Module (M_2). The potential difference at the Thermocouple reference junction output terminals is transmitted through copper conductors and applied across the input terminals of a Lowpass Filter. The conductors and the filter terminals comprise an interface between the Reference Junction and the Data Acquisition System, as shown in Figure 6-6.
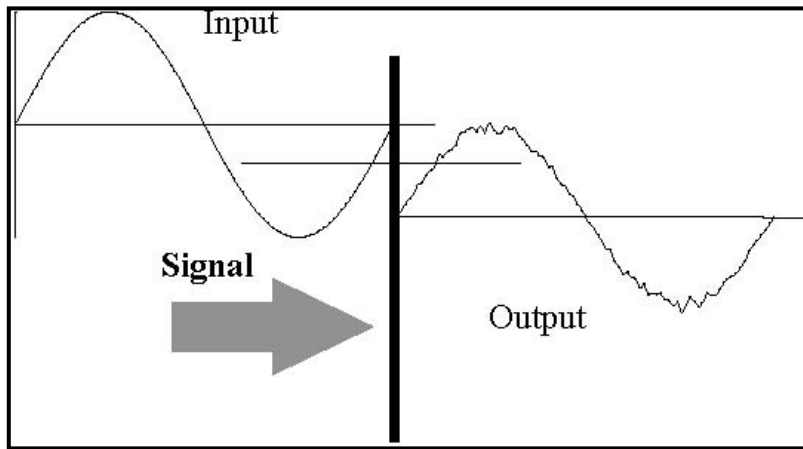
Figure 6-6. Schematic of Thermocouple - Lowpass Filter Interface.

The sources of error are

- Interface Loss
- Crosstalk
- Noise

6.3.2.1 Interface Loss. The voltage applied across the terminals of the Lowpass Filter suffers a drop due to the resistance of the connecting leads from the Reference Junction and of the Lowpass Filter contacts.

6.3.2.2 Crosstalk. Leakage currents between input filter terminals may alter the potential difference across the terminals.

6.3.2.3 Noise. Electromagnetic noise is a factor for the connecting leads, while both the connecting leads and the Lowpass Filter terminals are subject to thermal noise.

The output equation for the Interface1 module M_2 is

$$Y_2 = (1 + p_{21})(1 + p_{22})Y_1 + p_{23}$$

where

Parameter Name	Parameter Description	Nominal Value	Error Limits
Y_1	Output from module M_1		
p_{21}	Interface loss	0 µV	±0 µV
p_{22}	Crosstalk	0 µV	±0 µV
p_{23}	Noise	0 µV	±0 µV

Table 6-2. Parameters Used in Interface1 Module Output Equation

6.3.3 <u>Lowpass Filter Module (M_3)</u>. The potential difference that survives the Reference Junction - Filter interface is altered by the filter itself. The filter attenuates noise that may be present and provides a "cleaned up" potential difference to the system's amplifier. However, some noise gets through. Also, the filter attenuates the signal somewhat and itself generates a small noise component.
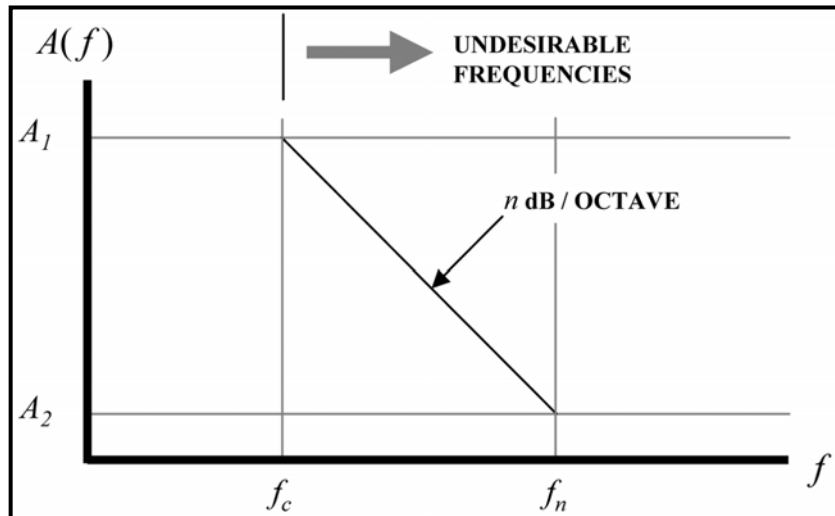
Figure 6-7. Amplitude as a Function of Frequency for a Typical Filter.

The sources of error are
- Signal Attenuation
- Noise
- Nonlinearity

6.3.3.1 <u>Signal Attenuation</u>. Although the filter is intended to attenuate unwanted noise, some signal attenuation also occurs.

6.3.3.2 <u>Noise</u>. Not all the input noise will be filtered out. The noise that remains will be attenuated by an amount that depends on the roll-off characteristics of the filter. These characteristics are usually assumed to be linear and are expressed in terms of dB per octave. Thermal noise is also generated within the filter itself.

6.3.3.3 <u>Nonlinearity</u>. The response of a filter over the range from its cutoff frequency, f_c, to its terminating frequency, f_n, is usually considered to be fairly linear. Departures from this assumed linearity constitute errors.

The output equation for the lowpass filter module M_3 is

$$Y_3 = \begin{cases} (1+p_{31})Y_2 + p_{32} & ,f \le f_c \\ (1+p_{31})Y_2 + \dfrac{p_{32} - (1+p_{31})Y_2}{p_{34} - p_{33}}(f - p_{33}) + p_{32} & ,f_c \le f \le f_n \\ p_{32} & ,f \ge f_n \end{cases}$$

where

	Table 6-3. Parameters Used In Low Pass Filter Module Output Equation		
Parameter Name	**Parameter Description**	**Nominal Value**	**Error Limits**
Y_2	Output from module M_2		
p_{31}	Signal attenuation	0 µV	±0.002 µV
p_{32}	Noise	0 µV	±0 µV
p_{33}	Cut-off frequency, f_c	100 Hz	±0 Hz
p_{34}	Maximum frequency output, f_n	N/A	±0 Hz

Because of the high cutoff frequency, only the upper equation for Y_3 applies.

6.3.4 <u>Interface2 Module (M_4)</u>. The potential difference output by the Lowpass Filter is fed to the Amplifier across an interface comprised of the leads from the Lowpass Filter and the input terminals of the Amplifier. The sources of error are

- Interface Loss
- Crosstalk
- Noise

6.3.4.1 <u>Interface Loss</u>. The voltage at the Amplifier terminals suffers a drop due to the resistance of the connecting leads from the Lowpass Filter and of the input terminal contacts.

6.3.4.2 <u>Crosstalk</u>. Leakage currents between input Amplifier terminals may cause a decrease in the potential difference across the terminals.

6.3.4.3 <u>Noise</u>. Electromagnetic noise is a factor for the connecting leads, while both the connecting leads and the Amplifier terminals are subject to thermal noise.

The output equation for the Interface2 module M4 is

$$Y_4 = (1 + p_{41})(1 - p_{42})Y_3 + p_{43}$$

where

Table 6-4. Parameters Used in Interface2 Module Output Equation

Parameter Name	Parameter Description	Nominal Value	Error Limits
Y_3	Output from module M_3		
p_{41}	Interface loss	0 µV	±0 µV
p_{42}	Crosstalk	0 µV	±0 µV
p_{43}	Noise	0 µV	±0 µV

6.3.5 <u>Amplifier Module (M5)</u>. The Amplifier increases the potential difference (and any noise received from the Lowpass Filter) and outputs the result to an A/D Converter (Figure 6-8).
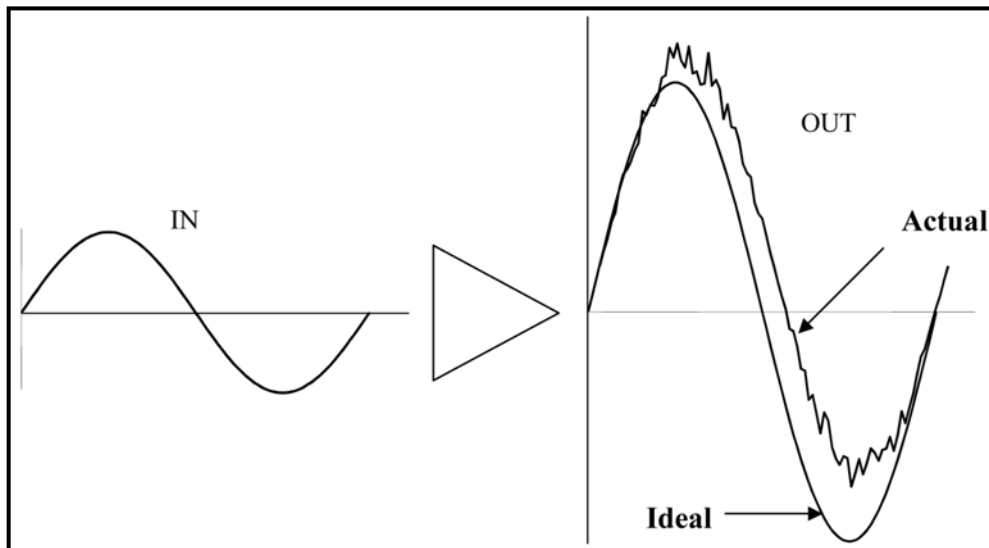
Figure 6-8. Amplifier Errors.

Several sources of error are present. Hysteresis causes an output time lag, common-mode voltage raises the zero reference, normal mode voltage stretches the waveform, while filtering compresses it. Riding on top is superimposed noise. Key error sources include:

- Gain
- Gain Instability
- Normal Mode Range
- Offset
- Nonlinearity
- Common Mode Rejection Ratio
- Noise

6.3.5.1 <u>Gain</u>. Amplifier gain is the ratio of the output signal voltage to the input signal voltage. Gain errors are those that lead to a uniform shift in expected Amplifier output vs. actual output. Gain errors are composed of inherent (systematic) errors and temperature induced (random and systematic) errors.

6.3.5.2 <u>Gain Instability</u>. If the Amplifier voltage gain is represented by G_V, its input resistance by R and its feedback resistance by R', then oscillations are possible when

$$\frac{RG_V}{R + R'} = \pi.$$

These oscillations appear as an instability in the amplifier gain.

6.3.5.3 <u>Normal Mode Voltage</u>. Normal mode voltages are differences in zero potential that occur when Amplifier input (signal) lines are not balanced. Normal mode voltages are essentially random in character.

6.3.5.4 <u>Offset</u>. Offset voltages and currents are applied to the Amplifier input terminals to compensate for systematically unbalanced input stages. The various parameters involved in offset compensation are the following:

6.3.5.4.1 <u>Input Bias Current</u>. A current supplied to compensate for unequal bias currents in input stages. Equal to one-half the sum of the currents entering the separate input terminals.

6.3.5.4.2 <u>Input Offset Current</u>. The difference between the separate currents entering the input terminals.

6.3.5.4.3 <u>Input Offset Current Drift</u>. The ratio of the change of input offset current to a change in temperature.

6.3.5.4.4 <u>Input Offset Voltage</u>. The voltage applied to achieve a zero Amplifier output when the input signal is zero.

6.3.5.4.5 <u>Input Offset Voltage Drift</u>. The ratio of the change of input offset voltage to a change in temperature.

6.3.5.4.6 <u>Output Offset Voltage</u>. The voltage across the Amplifier output terminals when the input terminals are grounded.

6.3.5.4.7 <u>Power Supply Rejection Ratio (PSRR)</u>. The ratio of the change in input offset voltage to the corresponding change in a given power supply voltage, with all other power supply voltages held fixed.

6.3.5.4.8 <u>Slew Rate</u>. The maximum time rate of change of the Amplifier output voltage under large-signal (usually square wave) conditions. Slew rate usually applies to the slower of the leading edge and trailing edge responses.

6.3.5.5 <u>Nonlinearity</u>. As with other components, actual Amplifier response may depart from the assumed output versus input curve. Nonlinearity error consists of the disagreement between the characteristic signature of an amplifier's response and its expected characteristic.

6.3.5.6 <u>Common Mode Rejection Ratio</u>. CMRR is the ratio of the amplifier signal voltage gain to the common mode voltage gain. CMRR is often used in estimating errors in amplifier output.

6.3.5.7 <u>Noise</u>. Noise generated within the Amplifier that enters the signal path causes errors in the output signal.

The output equation Amplifier module M_5 is

$$Y_5 = (p_{51} + p_{52} + p_{55})(Y_4 + p_{53} + p_{55} + p_{56}) + p_{57}$$

where

<div align="center">

Table 6-5. Parameters Used in Amplifier Module Output Equation

Parameter Name	Parameter Description	Nominal Value	Error Limits
Y_4	Output from module M_4		
p_{51}	Amplifier Gain	10	+ 1 % of Gain
p_{52}	Gain Instability	0 μV	± 0.05% of Gain
p_{53}	Normal Mode Voltage	0 μV	± 0 μV
p_{54}	Offset	0 μV	± 1.2 μV
p_{55}	Nonlinearity	0 μV	± 0.1% of Gain
p_{56}	Common Mode Reject Ratio	0 μV	± 0.002% of common mode input
p_{57}	Noise	0 μV	± 2.5 μV

</div>

6.3.6 <u>Interface3 Module (M_6)</u>. The amplified potential difference is applied across the A/D Converter input terminals. The interface between the Amplifier and the A/D Converter is prone to the following error sources:

- Interface Loss
- Crosstalk
- Noise

6.3.6.1 <u>Interface Loss</u>. The voltage at the A/D Converter terminals suffers a drop due to the resistance of the connecting leads from the Amplifier.

6.3.6.2 <u>Crosstalk</u>. Leakage between input A/D Converter may cause a decrease in the potential difference across the terminals.

6.3.6.3 <u>Noise</u>. Electromagnetic noise is a factor for the connecting leads, while both the connecting leads and the A/D Converter terminals are subject to thermal noise.

The output equation for the Interface3 module M_6 is

$$Y_6 = (1 + p_{61})(1 + p_{62})Y_5 + p_{63}$$

Where

Table 6-6. Parameters Used in Interface3 Module Output Equation			
Parameter Name	**Parameter Description**	**Nominal Value**	**Error Limits**
Y_5	Output from module M_5		
p_{61}	Interface loss	-0.01	+ 0.0002
p_{62}	Crosstalk	0 μV	± 0 μV
p_{63}	Noise	0 μV	± 0 μV

6.3.7 A/D Converter Module (M_7). In digitizing the analog signal, signal values are sampled as potential differences applied across a network of analog components. The network outputs a coded pulse consisting of ones and zeros. The location of these ones and zeros is a function of the input signal level and the response of the network to this signal level.

The errors in the digitizing process constitute a discrepancy between the waveform emerging from D/A conversion and the original input waveform (prior to A/D conversion). The sources of error are

- Sampling Rate Error
- Aperture Time Error
- Impulse Response Error
- Aliasing Error
- Digital Filtering Error
- Gain Error
- Noise
- Quantization Error

6.3.7.1 Sampling Rate Error. When a signal is sampled for digital data processing, it is represented by a set of discrete points. Later in the processing chain, these points need to be reconstituted into a signal again. When this is done, the reconstituted signal differs from the input signal by some amount.

What is done in the reconstitution process can be roughly described as an electronic or mathematical "connect the dots" procedure, as shown in Figure 6-9.
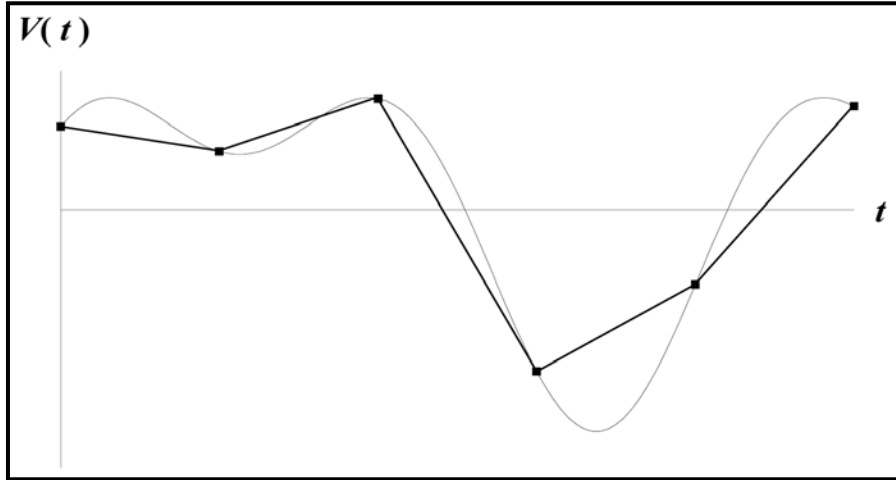
Figure 6-9. Sampling Rate Error.

Low-tech D/A converters reconstitute the signal using a set of straight lines from sampled point to sampled point. High-end D/A converters reconstitute the signal using involved mathematical operations utilizing sophisticated curve fitting and outlier detection techniques.

UncertaintyAnalyzer provides for both ends of the spectrum with a "Linear" curve fit model and a "Quadratic Spline" curve fit model. The latter employs a second degree curve fit, but improves on this by effectively doubling the sampling rate. The result is something that is more or less representative of high end D/A conversion.

For example, let us consider a signal represented by $f(t)$ that is digitized at discrete intervals Δt. The signal is later reconstructed into an approximation $\hat{f}(t)$. The signal $f(t)$ is composed of M components, each represented by a sine wave with angular frequency ω_m and relative phase φ_m.

$$f(t) = \sum_{m=0}^{M} a_m \sin(\omega_m t + \phi_m) \tag{6-17}$$

The sampling rate variance consists of the root mean square of $f(t) - \hat{f}(t)$ over a complete wave form.

$$\sigma^2 = \frac{1}{2\pi} \sum_{m=0}^{M} \int_{0}^{2\pi} \sigma_n^2 d\phi_m \tag{6-18}$$

where

$$\sigma_2^n = \frac{1}{\Delta} \int_{(n-1)\Delta}^{n\Delta} [f(t) - \hat{f}(t)]^2 \, dt. \tag{6-19}$$

6.3.7.1.1 <u>Linear Model</u>. The linear model is given by

$$\hat{f}(t) = f(0) + \frac{f(\Delta t) - f(0)}{\Delta} t \qquad (6\text{-}20)$$

where

$$f(0) = \sum_{m=0}^{M} a_m \sin\phi_m, \qquad (6\text{-}21)$$

and

$$f(\Delta t) = \sum_{m=0}^{M} a_m \sin(\omega_m \Delta t + \varphi_m). \qquad (6\text{-}22)$$

The sample rate variance for the linear model is

$$\sigma^2 = \sum_{m=0}^{M} a_m^2 \left(\langle I_1 \rangle_m - 2\langle I_2 \rangle_m + \langle I_3 \rangle_m \right), \qquad (6\text{-}23)$$

where

$$\langle I_1 \rangle_m = \frac{1}{2}, \qquad (6\text{-}24)$$

$$\langle I_2 \rangle_m = \frac{1 - \cos\omega_m t}{\left(\cos\omega_m \Delta t\right)^2}, \qquad (6\text{-}25)$$

and

$$\langle I_3 \rangle_m = \frac{1}{2}. \qquad (6\text{-}26)$$

6.3.7.1.2 <u>Quadratic Spline Model</u>. The quadratic spline model is given by

$$\hat{f}(t) = f(0) + b_1 t + b_2 t^2 \qquad (6\text{-}27)$$

where $f(0)$ is given above and

$$b_1 = \frac{1}{\Delta t}[4f(\Delta t/2) - f(\Delta t) - 3f(0)], \qquad (6\text{-}28)$$

and

$$b_2 = \frac{1}{\Delta t}[f(\Delta t) - 2f(\Delta t/2) + f(0).] \qquad (6\text{-}29)$$

The sampling rate variance for the quadratic model is also given by

$$\sigma^2 = \sum_{m=0}^{M} a_m^2 \left(\langle I_1 \rangle_m - 2\langle I_2 \rangle_m + \langle I_3 \rangle_m \right) \qquad (6\text{-}30)$$

where

$$\langle I_1 \rangle_m = \frac{1}{2},$$ (6-31)

$$\langle I_2 \rangle_m = \frac{1}{(\omega_m \Delta t)^2} \left[3 + \overline{c}_{m,1} - 4\overline{c}_{m,1/2} + \frac{4}{\omega_m \Delta t} \left(2\overline{s}_{m,1/2} - \overline{s}_{m,1} \right) \right],$$ (6-32)

and

$$\langle I_3 \rangle_m = \frac{1}{5} \left(2 + \frac{2}{3} \overline{c}_{m,1/2} - \frac{1}{6} \overline{c}_{m,1} \right),$$ (6-33)

The coefficients in equations (6-32) and (6-33) are

$$\overline{c}_{m,1} = \cos(\omega \Delta t),$$
$$\overline{c}_{m,1/2} = \cos(\omega \Delta t / 2),$$
$$\overline{s}_{m,1} = \sin(\omega \Delta t),$$
$$\overline{s}_{m,1/2} = \sin(\omega \Delta t / 2).$$

6.3.7.2 Aperture Time Error. A finite amount of time δt is required to sample the signal voltage V as shown in Figure 6-10. During this time, the signal value changes by an amount δV.

For a signal comprised of $N+1$ components of frequency ω_k and phase φ_k, the average error in δV is given by

$$\overline{\delta V} \cong \sum_{k=0}^{N} \frac{a_k}{24} (\omega_k \delta t)^2 \sin(\omega_k t + \varphi_k),$$ (6-34)

and

$$|\delta V|_{max} = \sum_{k=0}^{N} a_k \omega_k \delta t.$$ (6-35)
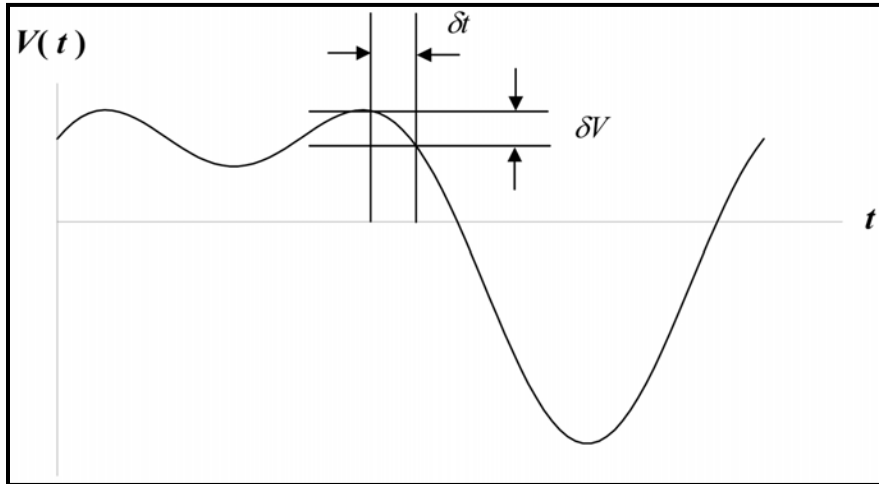
Figure 6-10. Aperture Time Error.

At this point, it is worth remembering that, in the measurement system, the sampled data will be quantized in binary code. As we will see later, these errors are on the order of

$$(\delta V)_{quantization} = \pm \frac{A}{2^{n+1}},$$ (6-36)

where n is the number of quantizing bits and A is the dynamic range of the quantizing apparatus.

If the quantization error is larger than the maximum aperture error, then the aperture error can be ignored. For example, for the composite sine wave signal considered here, aperture error can be ignored if

$$\sum_{k=0}^{N} a_k \omega_k \delta t < \frac{A}{2^n}, \quad or \quad \delta t < \frac{A}{\displaystyle\sum_{k=0}^{N} a_k \omega_k 2^n}.$$

6.3.7.3 <u>Impulse Response Error</u>. In sampling a signal, the sampling sensor must be able to respond to and recover from signal changes. If the rise times and recovery times of the sampling sensor are not negligible in comparison with the sampling aperture time, then impulse response errors occur.

For example, imagine that a signal is sampled in time increments of T over a sampling aperture of duration τ, as shown in Figure 6-11.
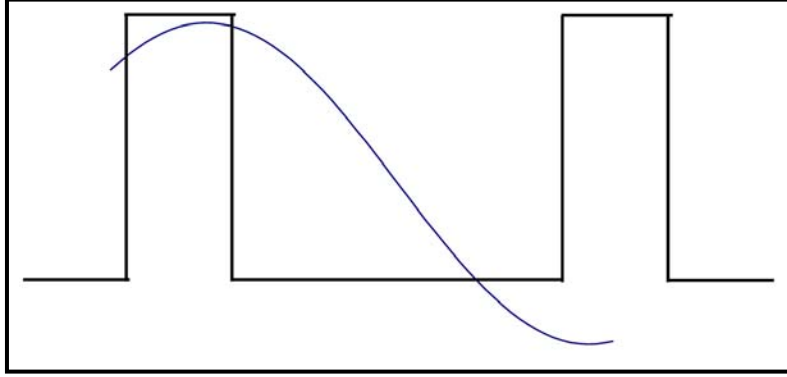
Figure 6-11. Impulse Error.

Assume that the response of the sensor *r(t)* to a signal value $V = a\sin(\omega t + \varphi)$ is governed by the equation:

$$\frac{dr}{dt} = \lambda(V - r). \tag{6-37}$$

We rewrite this equation in the following form to facilitate solution:

$$\frac{dr}{dt} + \lambda r = \lambda V. \tag{6-38}$$

In the region $0 < t < \tau$, the homogeneous solution to equation (1) is

$$r_h(t) = r_0 e^{-\lambda t}, \tag{6-39}$$

and the particular solution is

$$r_p(t) = b\sin(\omega t + \varphi) + c\cos(\omega t + \varphi). \tag{6-40}$$

Substituting $r(t) = r_h(t) + r_p(t)$ in equation (6-37) gives

$$(\omega b + \lambda c)\cos(\omega t + \varphi) + (\lambda b - \omega c)\sin(\omega t + \varphi) = \lambda a\sin(\omega t + \varphi) \tag{6-41}$$

so that

$$\omega b + \lambda c = 0$$
$$\lambda b - \omega c = \lambda a$$

and

$$b = \frac{1}{1 + q^2} a$$

$$c = -\frac{q}{1 + q^2} a,$$

where

$$q \equiv \omega / \lambda.$$

Since the value of φ will be an undetermined variable, we can set $r(0) = 0$ without loss of generality. This gives

$$r_0 = -\frac{a}{1+q^2}(\sin\varphi - q\cos\varphi), \tag{6-42}$$

and

$$r(\tau) = \frac{a}{1+q^2}[\sin(\omega t + \varphi) - q\cos(\omega t + \varphi) - (\sin\varphi - q\cos\varphi)e^{-\lambda t}]. \tag{6-43}$$

In the $\tau \le t \le T$ region, equation (6-37) becomes

$$\frac{dr}{dt} + \lambda r = 0, \tag{6-44}$$

with the solution

$$r(t) = r(\tau)e^{-\lambda(t-\tau)}, \tag{6-45}$$

where $r(\tau)$ is taken from equation (6-43).

In the $T \le t \le T + \tau$ region, the equation is again

$$\frac{dr}{dt} + \lambda r = \lambda V. \tag{6-46}$$

The homogeneous and particular solutions are, respectively,

$$r_h(t) = r_0 e^{-\lambda(t-T)} \tag{6-47}$$

$$r_p(t) = b\sin(\omega t + \varphi) + c\cos(\omega t + \varphi). \tag{6-48}$$

Substituting in equation (6-46) gives

$$r(t) = r_0 e^{-\lambda(t-T)} + \frac{a}{1+q^2}[\sin(\omega t + \varphi) - q\cos(\omega t + \varphi)]. \tag{6-49}$$

The initial condition is $r(T) = r(\tau)e^{-\lambda(T-t)}$ so that

$$r(\tau)e^{-\lambda(t-T)} = r_0 + \frac{a}{1+q^2}[\sin(\omega T + \varphi) - q\cos(\omega T + \varphi)] \tag{6-50}$$

and

$$r_0 = r(\tau)e^{-\lambda(t-T)} - \frac{a}{1+q^2}[\sin(\omega T + \varphi) - q\cos(\omega T + \varphi)]. \tag{6-51}$$

Substituting from equation (6-42) for $r(\tau)$ in equation (6-51) and plugging the result into equation (6-49) gives for $t - T + \tau$

$$r(T+\tau) = \left(\begin{array}{l} \left\{\dfrac{a}{1+q^2}\left[\sin(\omega\tau+\varphi)-q\cos(\omega\tau+\varphi)-(\sin\varphi-q\cos\varphi)e^{-\lambda\tau}\right]\right\}e^{-\lambda(T-\tau)} \\[4mm] -\dfrac{a}{1+q^2}\left[\sin(\omega T+\varphi)-q\cos(\omega T+\varphi)\right] \end{array}\right) \qquad (6\text{-}52)$$

$$+\dfrac{a}{1+q^2}\left\{\sin[\omega(T+\tau)+\varphi]-q\cos[\omega(T+\tau)+\varphi]\right\}$$

The difference between the input signal and the response at the point $t = T+\tau$ is the impulse response error:

$$\varepsilon_{ir} = r(T+\tau)-V(T+\tau). \qquad (6\text{-}53)$$

Since the phase φ is indeterminate, the impulse response variance is given by

$$\sigma_{ir}^2 = \frac{1}{2\pi}\int_0^{2\pi}\left\{r(T+\tau)-a\sin[\omega(T+\tau)+\varphi]\right\}^2 d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} r^2(T+\tau)d\varphi - \frac{a}{\pi}\int_0^{2\pi} r(T+\tau)\sin[\omega(T+\tau)+\varphi]d\varphi + \frac{a^2}{2}. \qquad (6\text{-}54)$$

Substituting equation (6-52) into this expression gives

$$\sigma_{ir}^2 = \frac{a^2}{2}\left\{1-\frac{1}{1+q^2}\left[1-e^{-2\lambda\tau}-e^{-2\lambda T}-e^{-2\lambda(T+\tau)}\right]\right\}. \qquad (6\text{-}55)$$

In general, this expression can be written

$$\sigma_{ir}^2 = \left\{V_{rms}^2 - \left(\frac{1}{2}\sum_{m=0}^{k}\frac{a_m^2}{1+q_m^2}\right)\left[1-e^{-2\lambda\tau}-e^{-2\lambda T}-e^{-2\lambda(T+\tau)}\right]\right\}. \qquad (6\text{-}56)$$

This form permits the evaluation of impulse response error for signals composed of multiple sine wave components:

$$V = \sum_{m=0}^{k} a_m \sin(\omega_m t + \varphi_m)$$

so that

$$V_{rms}^2 = \frac{1}{2}\sum_{m=0}^{k} a_m^2.$$

The parameter λ is obtained from the time it takes the sensor's response to reach $1/e$ of the input signal:

$$\begin{aligned} r(\tau_{1/e}) &= 1/e^{-\lambda\tau_{1/2}} \\ &= Ve^{-1} \end{aligned},$$

and

$$\lambda = \frac{1}{\tau_{1/e}}.$$

6.3.7.4 <u>Aliasing Error</u>. When two or more frequencies are sampled that are integral multiples of one another, these frequencies cannot be distinguished from one another. Aliasing error is avoided by setting the sampling rate higher than twice the frequency of the highest frequency component of the sampled signal and by employing low-pass filters in the input of the A/D converter.

6.3.7.5 <u>Digital Filtering Error</u>. Following A/D conversion, because of thermal noise in the filter and other factors, there is still a chance that undesirable high frequency contributions will be present. These contributions are ordinarily removed by lowpass digital filters.

Digital filtering of unwanted high frequencies is done by simply averaging the digitally encoded information over a set of sampled pulses. For example, suppose that the sampling frequency were double the highest usable frequency.

Since the sampling rate is twice the highest usable frequency, changes in amplitude that transpire between successive coded pulses are those that represent unwanted frequencies. By averaging the amplitudes of successive coded pulses or bitstreams, these frequencies are eliminated.

Unfortunately, the elimination of unwanted frequencies by digital filtering is not a free ride. The process introduces some error. Fortunately, for the frequencies involved in the present example, these errors are negligible and will not be covered here. For cases where these errors are significant, the reader is encouraged to survey the literature on digital filters.

6.3.7.6 <u>Gain Error</u>. One type of A/D Converter employs a ladder network of resistors. The configuration of the network is such that different signal levels cause different discrete responses. A major factor affecting the accuracy of these responses is the error in the value of the resistors in the network. This is because the voltage drop (negative gain) across each component resistor is a function of the signal level and the component's DC resistance.

6.3.7.7 <u>Noise</u>. As expected, stray voltages are sensed along with the signal voltage and contribute to the voltage level applied to the network. In addition, thermal fluctuations in components cause fluctuations in voltage drops.

6.3.7.8 <u>Quantization Error</u>. The potential drop (or lack of a potential drop) sensed across each element of the A/D Converter sensing network produces either a "1" or "0" to the converter. This response constitutes a "bit" in the binary code that represents the sampled value. The position of the bit in the code is determined by which network element originated it.

Even if no errors were present in sampling and sensing the input signal, errors would still be introduced by the discrete nature of the encoding process. Suppose, for example,

that the full scale signal level (dynamic range) of the A/D Converter is A volts. If n bits are used in the encoding process, then a voltage V can be resolved into 2^n discrete steps, each of size $A/2^n$. The error in the voltage V is thus

$$\varepsilon_V = V - m\frac{A}{2^n},$$
(6-57)

where m is some integer determined by the sensing function of the D/A Converter.

The uncertainty associated with each step is one-half the value of the magnitude of the step. Consequently, the uncertainty inherent in quantizing a voltage V is $(1/2)(A/2^n)$, or $A/2^{n+1}$. This is embodied in the expression

$$V_{quantized} = V_{sensed} \pm \frac{A}{2^{n+1}}.$$
(6-58)

With UncertaintyAnalyzer, the uncertainties due to sampling and other A/D conversion error sources are handled by the Digital Sampling Uncertainty Worksheet. If this worksheet is used, then the output equation for the D/A Converter module is simply

$$Y_7 = Y_6 + DS_7$$

where

Table 6-7. Parameters Used in Interface3 Module Output Equation			
Parameter Name	Parameter Description	Nominal Value	Error Limits
Y_6	Output from module M_6		
DS_7	Data Sampling Error		5.59724 μV

The uncertainty due to data sampling error was computed using UncertaintyAnalyzer's Digital Sampling Uncertainty Worksheet by entering the following information:

Input Signal Frequency:	10 Hz
Input Signal Amplitude:	(Computed by UncertaintyAnalyzer)
Sampling Rate:	200 Hz
Sampling Full Scale:	5 mV
Quantization Significant Bits:	16
Aperture Time:	20 μsec
Sensor Response Time Constant:	2.0 μsec
Signal Bandwidth:	10 Hz
Operating Temperature:	100 °C
Sensor Output Resistance:	100 kohm
Sensor RMS Current:	0.5 mA
Number of Sensor p-n Junctions:	1
Sensor Error Limits:	0.05% of input signal (99% confidence)

6.3.8 Data Processor Module (M_8). The quantized (digital) output from the A/D Converter is input to a Data Processor. Since the output is digital, the interface between the A/D Converter

and the Data Processor will be assumed to not constitute an error source. The Data Processor converts binary coded numbers to values, performs D/A curve fitting and applies any correction factors that may be appropriate. The sources of error are

- Correction Factor Error
- Data Reduction Error
- Decoding Error
- Computation Error

6.3.8.1 Correction Factor Error. The correction factor applied to the digitally encoded voltage difference attempts to correct for losses that occur between the Reference Junction and the Data Processor. Uncertainties in estimating these losses may lead to errors in the correction factors.

6.3.8.2 Data Reduction Error. In converting the corrected value for the voltage difference into a temperature difference, the Data Processor attempts to solve the equation

$$\Delta V = (a_1 - b_1)\Delta T + (a_2 - b_2)(\Delta T)^2 + (a_3 - b_3)(\Delta T)^3 + \cdots$$

In arriving at the solution, the series is truncated at some polynomial order. This truncation leads to a discrepancy between the solved-for temperature difference and the actual temperature difference.

For instance, suppose that the series is truncated to second order. Then the Data Processor solution for the temperature difference becomes

$$\Delta T = -\frac{1}{2}\left(\frac{a_1 - b_1}{a_2 - b_2}\right) + \sqrt{\frac{1}{4}\left(\frac{a_1 - b_1}{a_2 - b_2}\right)^2 + \frac{V_C^* - V_A^*}{a_2 - b_2}} + O3,$$

where the quantities V_C^* and V_A^* are corrected values for V_C and V_A, and O(3) represents the error due to neglecting third order and higher terms.

For the present example, data reduction error will be assumed to be nearly zero and will be ignored.

6.3.8.3 Decoding Error. The output of the Data Processor is a corrected result that is displayed as a decimal number. The following error sources are relevant in developing and displaying this number:

Suppose that the digital "resolution" of the binary encoded signal is $A/2^n$. Suppose further that the full-scale value Data Processor readout is S and that m digits are displayed. Then the resolution of the decimal display of the Data Processor is $S/10^m$. Another way of saying this is that the input to the Data Processor is a multiple of steps of size

$$h_b = \frac{A}{2^n},$$

while the decimal encoded display is presented in steps of size

$$h_d = \frac{S}{10^m}.$$

What this means is that a binary encoding of a voltage V into a representation

$$V' = 2^x h_b$$

will be translated into a decimal representation

$$V'' = 10^y h_d$$

where x and y are integers.

The quantization error that results from expressing an analog value first as a binary coded value and second as a decimal coded value is the sum of these two errors:

$$\text{Quantization Error} = \pm \frac{(h_b + h_d)}{2} = \pm \frac{A}{2^{n+1}} \pm \frac{S/2}{10^m}$$

6.3.8.4 <u>Computation Error</u>. Converting from a digital representation of an input signal to an analog (or at least decimal) representation involves a substantial amount of numerical computation. Each time a computation is made, some round off error is introduced. These errors accumulate through the computing process.

The computation error uncertainty was computed using UncertaintyAnalyzer's Computation Error Uncertainty Worksheet by entering the following information:

Error Source Description	Computed or Measured Value	Approx. No. Calcs	Decimal Digits	Source Coefficient
Curve Fit	8800	2000	8	1.0
Fine Tuning	25	100	8	1.0

The output equation for the data processor module M_8 is

$$Y_8 = p_{81} p_{83} Y_7 + p_{82} + CE_8$$

where

Table 6-8. Parameters Used in Data Processor Module Output Equation			
Parameter Name	Parameter Description	Nominal Value	Error Limits
Y_7	Output from module M_7		
p_{81}	Voltage to temperature conversion factor	0.0245 °C/µV	+ 0.0012 °C/µV
p_{82}	Decoding Error	0 °C	+ 0.10 °C
p_{83}	Correction factor	0.101 °C	+ 0.000002 °C
CE_8	Computation Error		1.323E-7 °C

6.4 Module Uncertainties

The next step in the system uncertainty analysis procedure is to input information about the parameters for each module and to estimate the uncertainties in the errors in these parameters. This can be done by entering the requisite data for each parameter directly into the Module Parameters table in the Module Analysis Screen or using the associated Error Source Worksheets.

6.4.1 Module Parameters Table. The Module Parameters table is used to enter and store the parameter information for each module. For example, the Thermocouple module output equation has six parameters, as listed in Table 6-1.

As previously discussed, the system input (X) is defined via the Subject Parameter Bias Uncertainty Worksheet. Brief descriptions of the module parameters are entered into the Description column of the table. For the thermocouple module, Sensitivity would be entered into the Description field for parameter p11, and so on.

The standard uncertainty for a parameter is either computed from user specified ± Error Limits and % Confidence, or entered directly. The probability that the error in the module parameter will be contained within the specified ± error limits is entered into the % Confidence column of the table.

UncertaintyAnalyzer assumes that the uncertainties in the parameter errors are normally distributed unless the associated degrees of freedom are less than infinite or the % Confidence is 100%. If the degrees of freedom are less than infinite and the % Confidence is less than 100%, the Student's t distribution is used. If the % Confidence is 100%, then the Uniform distribution is used and the degrees of freedom are assumed to be infinite.

6.4.2 Error Source Worksheets. Alternatively, the module parameter uncertainties can be estimated using the Error Source Worksheets accessed by clicking the Edit button to the left of the desired parameter name. The Error Source Worksheets provide a useful tool for making both Type A and Type B uncertainty estimates.

The Type A Uncertainty portion of the worksheet can be used to statistically estimate the uncertainty in the parameter error from measured values or deviations from nominal that are entered into the data entry table.

The Type B Uncertainty portion of the worksheet can be used to develop a heuristic uncertainty estimate from bounding values, or ± limits, that are expected to contain the error with some specified probability or confidence level.

UncertaintyAnalyzer assumes infinite degrees of freedom for the Type B uncertainty estimate unless otherwise specified by the user. Since, the degrees of freedom quantifies the amount of knowledge available for making the uncertainty estimate, an infinite degrees of freedom signifies complete certainty, i.e., zero uncertainty.

In most cases, it would not be realistic to assume infinite knowledge about the uncertainty. The **Type B Degrees of Freedom Calculator** can be used to provide additional information about the uncertainty in the ± limits and associated confidence level.

The statistical distribution for errors whose uncertainties are arrived at as a result of a Type B analysis can be selected from the drop-down Distribution list. In most cases, the Normal distribution will be applicable. For cases where it is suspected or known that errors are bounded by finite limits, other distributions may be more applicable. In cases where the degrees of freedom are finite, the Student's t distribution is automatically selected. The distribution options available from the Error Source Worksheet include:

- Normal
- Quadratic
- Cosine
- U-Shaped

- Uniform
- Triangular
- Student's t

6.4.3 <u>Sensitivity Coefficients</u>. UncertaintyAnalyzer automatically computes the partial derivatives of the selected module output equation with respect to each module parameter using the method previously described in Section 5.4.1. If a coefficient equation has been entered for a given module parameter, then UncertaintyAnalyzer computes the Sensitivity Coefficient from this equation and enters its value into the Module Parameters table.

6.4.4 <u>Component Uncertainty</u>. The component uncertainty for a module parameter is the product of the parameter's sensitivity coefficient and standard uncertainty. It is a weighted value that reflects the contribution of the uncertainty in the parameter error to the overall uncertainty in the module output value. UncertaintyAnalyzer automatically computes component uncertainties for all variables in the Module Parameters table.

6.4.5 <u>Total Module Uncertainty</u>. After the uncertainties have been estimated or entered for each module parameter and the associated sensitivity coefficients and component uncertainties have been computed, selecting the AutoCalc All icon on the System Model Screen toolbar computes the module output value and total uncertainty.

If the error sources selected for inclusion in the module analysis are independent (i.e., uncorrelated), then the error source uncertainties are combined in a root-sum-square (RSS) manner to obtain the Total Module Uncertainty. Otherwise, the Total Module Uncertainty will reflect any correlations between error sources specified by the user via the Correlation Analysis Screen.

The System Module Analysis table provides a summary table for the system modules. The summary information for our Digital Temperature Measurement System analysis is shown below.

Module Name	Input	Output	Uncertainty	Units	Coefficient
Thermocouple Sensor	22.0	888.9	5.63	µV	0
Thermocouple - Filter Interface	888.9	888.9	5.63	µV	0
Lowpass Filter	888.9	888.9	5.7	µV	0
Filter - Amplifier Interface	888.9	888.9	5.7	µV	0
Amplifier	888.9	8,889.2	73.2	µV	0
Amplifier - A/D Converter	8,889.2	8,889.3	72.8	µV	0
A/D Converter	8,889.3	8,889.3	73	µV	0
Data Processor	8,889.3	21.78	0.459	°C	1.000

The last column of the table contains a coefficient indicating the contribution made by a given module to the total system output. For this example, all modules should have a coefficient equal to 0.0, except the last module which has a value of 1.0.

6.5 System Output Uncertainty

The system output value is equal to the computed output for the final module in the series. Each module output calculation is dependent upon its preceding module. Consequently, the system

output value will change until all necessary parameter information has been entered and associated calculations completed for each module.

Similarly, the total system uncertainty is equal to the output uncertainty for the final module. The associated degrees of freedom for the total system uncertainty is also equal to the degrees of freedom for the final module output uncertainty. The analysis results for our Digital Temperature Measurement System analysis is shown below.

System Input	22.0
System Output	21.8 °C
Total System Uncertainty	0.46 deg C
Degrees of Freedom	infinite

The Total System Uncertainty of 0.46 deg C is the uncertainty of the system output at 22 °C. We can use this uncertainty value to tolerance the Digital Temperature Measurement System. The tolerance limits for the system are computed from the degrees of freedom and user specified confidence level.

For example, if we wanted to tolerance the system to 95% confidence limits, the corresponding tolerance limits would be ±0.90 deg C. Alternatively, if we wanted to tolerance the system using the NIST convention of multiplying the uncertainty by a factor of 2, the tolerance limits would be ±0.92 deg C with a confidence level of 95.45%.

Either approach is acceptable. For this example, we will use the NIST approach, writing as a specification for the measurements in the 22 °C range

$$T_{out} = T_{in} \pm 0.91\,^{\circ}\text{C, coverage factor of 2.00.}$$

The amount of time that this tolerance is applicable depends on the stability of the various parameters of the system.

We can repeat the analysis process for other input temperatures. For example, suppose that we were to repeat the our analysis for input temperatures of 20 °C, 40 °C, 60 °C, 80 °C, and 100 °C. All parameter specifications are the same as previously stated, except the thermocouple sensitivity (Seebeck Coefficient), which is specified for each temperature as follows:

Input Temp.	Sensitivity (p11)	± Error Limits	% Confidence
20 °C	40.313 µV/°C	0.076 µV/°C	99.0
40 °C	40.977 µV/°C	0.056 µV/°C	99.0
60 °C	41.407 µV/°C	0.029 µV/°C	99.0
80 °C	41.541 µV/°C	0.010 µV/°C	99.0
100 °C	41.371 µV/°C	0.031 µV/°C	99.0

The analysis results are summarized below.

Input Temp.	Output Temp.	Output Uncertainty	± Tolerance Limits
20 °C	19.8 °C	0.42 °C	0.84 °C
40 °C	40.2 °C	0.81 °C	1.62 °C
60 °C	60.9 °C	1.22 °C	2.44 °C
80 °C	81.4 °C	1.63 °C	3.26 °C
100 °C	101.3 °C	2.02 °C	4.04 °C

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CHAPTER 7

UNCERTAINTY GROWTH ESTIMATION

7.0 General

The error or bias in a subject parameter may grow with time or may remain constant. In some cases, it may even shrink. The uncertainty in this error, however, *always* grows with time since measurement or calibration. This is the **fundamental postulate** of uncertainty growth. This chapter discusses the methodology used to project uncertainty growth in the subject parameter bias.[15]

Figure 7-1 illustrates uncertainty growth over time for a typical measurement attribute or parameter. The sequence shows the statistical distribution at three different times, with the uncertainty growth reflected in the spreads in the curves. The out-of-tolerance probabilities at the different times are represented by the shaded areas under the curves.
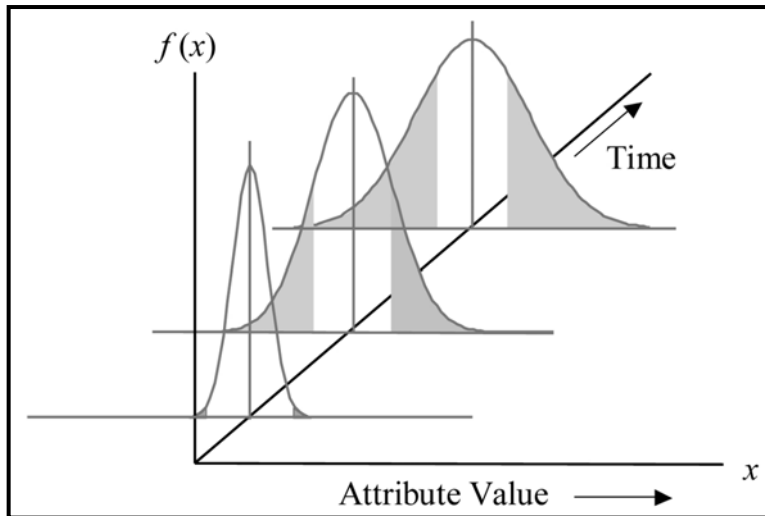
Figure 7-1. Measurement Uncertainty Growth.

The growth in uncertainty over time corresponds to an increase in out-of-tolerance probability over time. Conversely, it corresponds to a decrease in in-tolerance probability, or measurement reliability over time. Plotting this quantity versus time, as shown in Figure 7-2, suggests that measurement reliability can be modeled by a time-varying function. Once this function is determined, the uncertainty in the bias of a parameter can be computed as a function of time.

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[15] The methodology presented herein was developed by Dr. H. Castrup of Integrated Sciences Group (see References).
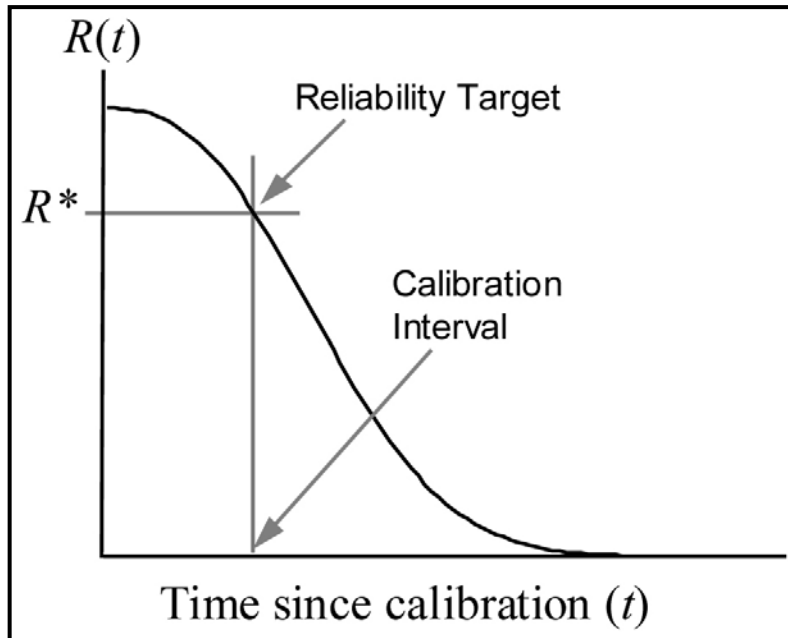
Figure 7-2. Measurement Reliability versus Time.

7.1 Definitions

7.1.1 <u>AOP Reliability</u>. The in-tolerance probability for a parameter averaged over its calibration or test interval. The average-over-period (AOP) measurement reliability is often used to represent the in-tolerance probability of a parameter for a measuring item whose usage demand is random over its test or calibration interval.

7.1.2 <u>Bias Reference</u>. The time to which the in-tolerance probability of a parameter is referenced.

7.1.3 <u>BOP Reliability</u>. The in-tolerance probability for a parameter at the start or beginning-of period (BOP) of its calibration or test interval.

7.1.4 <u>Calibration</u>. With reference to industrial and scientific instruments: 1) To adjust the output of a device, to bring it to a desired value, within a specified tolerance, for a particular value of an input; 2) to ascertain the error in the output of a device by measuring or comparing against a standard.

7.1.5 <u>Calibration Interval</u>. The elapsed time between successive calibrations of a given equipment parameter or attribute.

7.1.6 <u>EOP Reliability</u>. The in-tolerance probability for a parameter at the end of its calibration or test interval (i.e., end-of-period).

7.1.7 <u>Reliability Model</u>. In uncertainty analysis measurement decision risk analysis and calibration interval analysis, a reliability model is a mathematical function whose characteristics are based on a parameter's calibration history. Used to project uncertainty growth over time.

7.1.8 <u>Uncertainty Growth</u>. The increase in the uncertainty in the value of a parameter or other attribute over the time elapsed since measurement.

7.2 Basic Methodology

UncertaintyAnalyzer uses the combined uncertainty at the time of measurement and the reliability model information to estimate the uncertainty in the subject parameter value at the specified time since measurement.

The uncertainty, u(t), in the value of a subject parameter at time t elapsed since measurement ($t = 0$) is computed using the value of the initial measurement uncertainty, $u(0)$, and the reliability model for the parameter population. The basic concept is an extension of the ergodic theorem that states that the distribution of an infinite population of values at equilibrium is identical to the distribution of values attained by a single member sampled an infinite number of times.

The reliability model predicts the in-tolerance probability for the subject parameter population as a function of time elapsed since measurement. It can be thought of as a function that quantifies the *stability* of the population. In this view, we begin with a population in- tolerance probability at time $t = 0$ (immediately following measurement) and extrapolate to the in-tolerance probability at time $t > 0$.

The reliability (in-tolerance probability) of the subject parameter at time t is related to the parameter's uncertainty according to

$$R(t) = \int_{-L_1}^{L_2} f[x|u(t)]dx, \tag{7-1}$$

where $f[x \mid u(t)]$ is the probability density function for the parameter and x represents deviations from the parameter's design or nominal value. For purposes of discussion, we will assume for the moment that the subject parameter pdf is given by

$$f[x|u(t)] = \frac{1}{\sqrt{2\pi}u(t)} e^{-(x-\mu)/2u^2(t)}. \tag{7-2}$$

We state that at a given time t, the subject parameter's true deviation from nominal is given by the relation

$$\mu(t) = \mu_0 + b(t), \tag{7-3}$$

where $b(0)=0$.

The relationship between L_1, L_2 and μ is shown in Figure 7-3, along with the distribution of the population of biases for the subject parameter of interest.
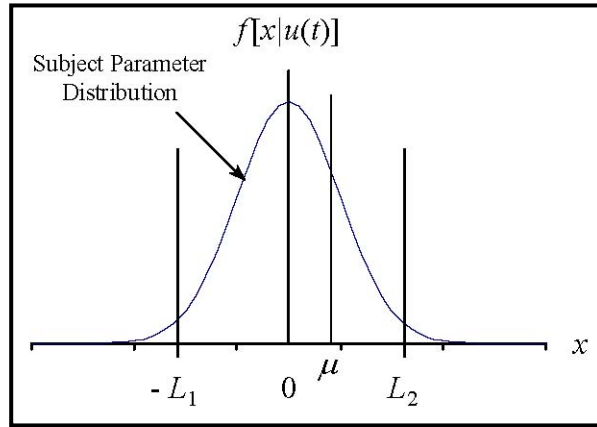
Figure 7-3. Subject Parameter Distribution.

At the time of measurement ($t = 0$), we estimate a value for μ_0 and label the uncertainty in this estimate $u(0)$. The remainder of this note discusses a method for calculating $u(t)$, given $u(0)$.

7.3 Projected Uncertainty

If we had at our disposal the reliability model for the individual measured parameter, given its initial uncertainty, we could obtain the uncertainty $u(t)$ in equation (7-1) directly by iteration or other means. However, we usually have information that relates only to the characteristics of the reliability model to which the subject parameter belongs.

We apply the reliability model for the population to the individual parameter under consideration. In UncertaintyAnalyzer, using a population reliability model to estimate uncertainty growth for a parameter employs the following set of premises:

1. The result of a parameter measurement is an estimate of a parameter's value or bias. This result is accompanied by an estimate of the uncertainty in the parameter's bias.

2. The uncertainty of the measured parameter's bias or value at time $t = 0$ (immediately following measurement) is the estimated uncertainty of the measurement process.

3. The bias or value of the measured parameter is either normally distributed or t-distributed around the measurement result.

4. The stability of the parameter is equated to the stability of its population. This stability is represented by the populations' reliability model.

5. Therefore, the uncertainty in the parameter's value or bias grows from its value at t = 0 in accordance with the reliability model of the parameter's population.

The reliability function at time t elapsed since measurement is given by

$$R(t) = \int_A f[x|\mu(t), u(t)]dx, \tag{7-4}$$

where A is the "acceptance" region for values of x. For example, if the parameter is two-sided, with tolerance limits $-L_1$ and L_2, then A is comprised of all values of x such that $-L_1 \leq x \leq L_2$, and $R(t)$ is as given in equation (7-1).

The reliability function at time $t = 0$ is likewise given by

$$R(0) = \int_A f[x|\mu_0, u(0)]dx. \tag{7-5}$$

The first step in estimating $u(t)$ is to form the ratio

$$\frac{R(t)}{R(0)} = \frac{\int_A f[x|\mu(t), u(t)]dx}{\int_A f[x|\mu_0, u(0)]dx}. \tag{7-6}$$

Next, we employ reliabilities $R(t)$ and $R(0)$, computed for the population of items represented by the distribution for $\underline{x}$ and solve for u($\underline{t}$). However, for consistency on both sides of the equation, we need to average the numerator and denominator on the right hand side over μ_0 and μ, respectively. Since these quantities are errors in x, their expectation value is zero, and we have the general relation

$$\frac{R(t)}{R(0)} = \frac{\int_A f[x|u(t)]dx}{\int_A f[x|u(0)]dx}. \tag{7-7}$$

7.3.1 <u>Normally Distributed Parameter Values</u>. Errors or biases are assumed to follow the normal distribution in cases where the degrees of freedom associated with the uncertainty estimate is infinite.

7.3.1.1 <u>General Two-Sided Cases</u>. For parameters with two-sided tolerance limits, the reliability function at $t = 0$ is given by

$$R(0) = \frac{1}{\sqrt{2\pi}u(0)} \int_{-L_1}^{L_2} e^{-x^2/2u_0^2} dx$$
$$= \Phi\left(\frac{L_1}{u_0}\right) + \left(\frac{L_2}{u_0}\right) - 1, \tag{7-8}$$

where

$$u_0 \equiv u(0).$$

The reliability at time $t > 0$ is given by

$$R(t) = \Phi\left[\frac{L_1}{u(t)}\right] + \Phi\left[\frac{L_2}{u(t)}\right] - 1. \tag{7-9}$$

To solve for u(t), we take ratio of equation (7-9) to equation (7-8)

$$\frac{R(t)}{R(0)} = \frac{\Phi\left[\dfrac{L_1}{u(t)}\right] + \Phi\left[\dfrac{L_2}{u(t)}\right] - 1}{\Phi\left(\dfrac{L_1}{u_0}\right) + \left(\dfrac{L_2}{u_0}\right) - 1}, \tag{7-10}$$

where $R(0)$ and $R(t)$ are determined using the reliability model. In this equation, u_0, $R(t)$ and $R(0)$ are known.

To put equation (7-10) in a form that amenable to solution, we write

$$\Phi\left[\frac{L_1}{u(t)}\right] + \Phi\left[\frac{L_2}{u(t)}\right] = 1 + \left[\Phi\left(\frac{L_1}{u_0}\right) + \Phi\left(\frac{L_2}{u_0}\right) - 1\right]\frac{R(t)}{R(0)}. \tag{7-11}$$

The uncertainty $u(t)$ is obtained through iteration. The iteration process employs the Newton-Raphson method. In this method, a function H and its derivative H' are defined according to

$$H = \varphi(x) - \rho\varphi(x_0)$$

and

$$H' = \varphi'(x),$$

where

$$\varphi(x) = \Phi(L_1 x) + \Phi(L_2 x) - 1,$$

$$\varphi'(x) = \frac{1}{\sqrt{2\pi}}\left[L_1 e^{-(L_1+\mu_0)^2 x^2/2} + L_2 e^{(L_2-\mu_0)^2 x^2/2}\right],$$

$$x \equiv 1/u(t),$$

$$x_0 \equiv 1/u_0,$$

$$\rho \equiv R(t)/R(0).$$

The iterations begin with a starting value for x of

$$x^{(0)} = \frac{1}{L}\Phi^{-1}\left\{\frac{1 + \rho\varphi(x_0)}{2}\right\},$$

where $L = (L_1 + L_2)/2$. Starting with this value, the iteration proceeds according to

$$x^{(r+1)} = x^{(r)} - H/H'.$$

The iterations stop when the magnitude of H/H' shrinks to some predetermined value $\varepsilon \ll 1$.

7.3.1.2 <u>Symmetric Two-Sided Cases</u>. In cases where $L_1 = L_2 = L$, equation (7-11) becomes

$$2\Phi\left[\frac{L}{u(t)}\right] = 1 + \left\{2\Phi\left(\frac{L}{u_0}\right) - 1\right\}\rho, \tag{7-12}$$

and

$$u(t) = \frac{L}{\Phi^{-1}\left(\frac{1}{2}\{1+[2\Phi(L/u_0)]-1\}\rho\right)}.$$ (7-13)

7.3.1.3 <u>Single-Sided Cases</u>. In cases where tolerances are single-sided, $u(t)$ can be determined without iteration. In these cases, either L_1 or L_2 is infinite, and equation (7-11) becomes

$$\Phi\left[\frac{L}{u(t)}\right] = \Phi\left(\frac{L}{u_o}\right)\rho,$$ (7-14)

where L is equal to $-L_1$ for single-sided lower cases and equal to L_2 for single-sided upper cases. Solving for $u(t)$ yields

$$u(t) = \frac{L}{\Phi^{-1}\left\{\Phi\left(\frac{L}{u_0}\right)\rho\right\}}.$$ (7-15)

For example, if $R(t) = R_0 e^{-\lambda t}$, then

$$u(t) = \frac{L}{\Phi^{-1}\left\{\Phi(L/u_0)e^{-\lambda t}\right\}}.$$

7.3.2 <u>Student's t-Distributed Parameter Values</u>. For uncertainty estimates $u(0)$ whose degrees of freedom is finite, the applicable distribution is the Student's t distribution.

7.3.2.1 General Tw0-Sided Cases. The treatment is the same as for normally distributed values, except that the distribution in equation (7-8) is the t distribution.

$$R(t) = C_v \int_{-L_1/u(t)}^{L_2/u(t)} \left(1+x^2/v\right)^{-(v+1)/2} dx,$$ (7-16)

where

$$C_v = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \frac{1}{\sqrt{\pi v}},$$ (7-17)

and v the degrees of freedom for the uncertainty estimate u_0.

   In evaluating equation (7-16), we define a function F according to

$$F_v(K) = C_v \int_{-\infty}^{K} \left(1+x^2/v\right)^{-(v+1)/2} dx.$$ (7-18)

We will also employ the function $F_v(-K)$, which, after a little manipulation, can be written

$$F_v(-K) = 1 - F_v(K). \tag{7-19}$$

The Student's t distribution, denoted here by the function G_v is usually expressed as the probability of lying within symmetric limits $\pm K$

$$G_v(K) = C_v \int_{-K}^{K} \left(1 + x^2/v\right)^{-(v+1)/2} dx \tag{7-20}$$
$$= 2F_v(K) - 1,$$

so that

$$F_v(K) = \frac{1}{2}\left[1 + G_v(K)\right]. \tag{7-21}$$

Using equations (7-20) and (7-21) in equation (7-16) gives

$$R(t) = F_v\left[\frac{L_2}{u(t)}\right] + F_v\left[\frac{L_1}{u(t)}\right] - 1 \tag{7-22}$$
$$= \frac{1}{2}\left\{G_v\left[\frac{L_2}{u(t)}\right] + G_v\left[\frac{L_1}{u(t)}\right]\right\}.$$

Note that, $G_v(\infty) = 1$, so that single-sided cases are described by

$$R(t) = \frac{1}{2}\left\{G_v\left[\frac{L}{u(t)}\right] + 1\right\}, \text{ (single-sided cases)}. \tag{7-23}$$

As in equation (7-10), we have

$$\rho = \frac{R(t)}{R(0)} = \frac{G_v\left[\dfrac{L_1}{u(t)}\right] + G_v\left[\dfrac{L_2}{u(t)}\right]}{G_v\left(\dfrac{L_1}{u_0}\right) + G_v\left(\dfrac{L_2}{u_0}\right)},$$

and

$$G_v\left[\frac{L_1}{u(t)}\right] + G_v\left[\frac{L_2}{u(t)}\right] = \left\{G_v\left(\frac{L_1}{u_0}\right) + G_v\left(\frac{L_2}{u_0}\right)\right\}\rho. \tag{7-24}$$

The Newton-Raphson solution for $u(t)$ is obtained analogously to the solution for normally distributed cases. For Student's t distributions,

$$H = \varphi_v(x) - \rho\varphi_v(x_0)$$

and

$$H' = \varphi'(x),$$

where

$$\varphi(x) = G_\nu(L_1 x) + G_\nu(L_2 x),$$

$$x \equiv 1/u(t),$$

$$x_0 \equiv 1/u_0,$$

and

$$\varphi'(x) = 2C_\nu \left[\left(1 + L_1^2 x^2 / \nu\right)^{-(\nu+1)/2} + \left(1 + L_2^2 x^2 / \nu\right)^{-(\nu+1)/2} \right].$$

The initial value for x is set at

$$x^{(0)} = \frac{1}{\mathcal{L}} G_\nu^{-1} \left[G_\nu(\mathcal{L}/u_0)\rho \right],$$

or

$$x^{(0)} = \frac{1}{\mathcal{L}} F_\nu^{-1} \left[\frac{1 + G_\nu(\mathcal{L}/u_0)\rho}{2} \right],$$

where $\mathcal{L}$ is defined as before, and F_ν^{-1} is the inverse t distribution function.

7.3.2.2 <u>Symmetric Two-Sided Cases</u>. In cases where $L_1 = L_2 = L$, equation (7-24) becomes

$$2F_\nu \left[\frac{L}{u(t)} \right] - 1 = G_\nu \left(\frac{L}{u_0} \right)\rho , \tag{7-25}$$

and

$$u(t) = \frac{L}{F_\nu^{-1} \left\{ \frac{1}{2} \left[1 + \rho G_\nu \left(\frac{L}{u_o} \right) \right] \right\}} . \tag{7-26}$$

7.3.2.3 <u>Single-Sided Cases</u>. In cases where tolerances are single-sided, $u(t)$ can be determined analytically. In these cases, either L_1 or L_2 in infinite, and equation (7-24) becomes

$$2F_\nu \left[\frac{L}{u(t)} \right] = \left[1 + G_\nu \left(\frac{L}{u_0} \right) \right]\rho , \tag{7-27}$$

where L is equal to L_1 for single-sided lower cases and equal to L_2 for single-sided upper cases. Solving for $u(t)$ yields

$$u(t) = \frac{L}{F_\nu^{-1} \left\{ \frac{\rho}{2} \left[1 + G_\nu \left(\frac{L}{u_0} \right) \right] \right\}} . \tag{7-28}$$

7.4 Reliability Models

In the uncertainty growth projection process, we utilize information about the calibration history of the subject parameter to develop a reliability model. This reliability model provides a means for determining how the subject parameter bias uncertainty grows with time since calibration. This means that the uncertainty surrounding the measured value we report will increase with time until the next calibration.

If we have access to a reliability modeling application, we can identify the appropriate reliability model and acquire the model's characteristics and enter this information into UncertaintyAnalyzer's Subject Parameter Reliability Model Worksheet.

Alternatively, we can enter an elapsed time, a beginning-of-period (BOP) reliability and an end-of-period (EOP) reliability for the calibration interval. For certain models, we must also enter an average-over-period (AOP) reliability. These values apply to the subject parameter's population and are based on service history records or engineering knowledge.

The Subject Parameter Reliability Model Worksheet has eight reliability models to choose. Each model is defined by a mathematical equation with characteristic coefficients. An applicable reliability model must be chosen based on knowledge about the stability of the subject parameter over time.

The application of each of the reliability models available in UncertaintyAnalyzer is described below along with information needed to implement each of them.

7.4.1 <u>Exponential Model</u>. The exponential reliability model is defined by the mathematical equation

$$R(t) = ae^{-bt} \qquad (7\text{-}29)$$

where $R(t)$ is the in-tolerance probability at time t and a and b are the model coefficients.

The exponential model is useful for parameters whose failure probability is not a function of time interval T, beginning at some time t, is the same as the probability of going out-of-tolerance in the same time interval T, beginning at some other time t'.

To implement the exponential model you need to know either of the following:

1. The value of the model coefficients, a and b.

2. The beginning of period (BOP) in-tolerance probability and the end of period (EOP) in-tolerance probability.

7.4.2 <u>Mixed Exponential Model</u>. The mixed exponential reliability model is defined by the mathematical equation

$$R(t) = \frac{1}{\left(1 + \dfrac{at}{b}\right)^{b}} \qquad (7\text{-}30)$$

where $R(t)$ is the in-tolerance probability at time t and a and b are the model coefficients.

The mixed exponential model is useful for parameters whose out-of-tolerance behavior depends on a number of constituent parameters, each of which can be modeled with the exponential model.

To implement the mixed exponential model you need to know either of the following:

1. The value of the model coefficients, a and b.

2. The BOP and EOP in-tolerance probabilities.

7.4.3 <u>Weibull Model</u>. The Weibull reliability model is defined by the mathematical equation

$$R(t) = ae^{-(bt)^c} \tag{7-31}$$

where $R(t)$ is the in-tolerance probability at time t and a, b and c are the model coefficients.

The Weibull model is useful for parameters that go out-of-tolerance as a result of gradual wear or decay.

To implement the Weibull model you need to know either of the following:

1. The value of the model coefficients, a, b and c.

2. The BOP and EOP in-tolerance probabilities and the average-over period (AOP) in-tolerance probability.

7.4.4 <u>Gamma Model</u>. The gamma reliability model is defined by the mathematical equation

$$R(t) = \frac{ae^{-bt}}{1 + bt + \dfrac{(bt)^2}{2} + \dfrac{(bt)^3}{6}} \tag{7-32}$$

where $R(t)$ is the in-tolerance probability at time t and a and b are the model coefficients.

The gamma model is useful for parameters that go out-of-tolerance in response to some number of events, such as being activated and deactivated.

To implement the gamma model you need to know either of the following:

1. The value of the model coefficients, a and b.

2. The BOP and EOP in-tolerance probabilities.

7.4.5 <u>Mortality Drift Model</u>. The mortality drift reliability model is defined by the mathematical equation

$$R(t) = ae^{-(bt+ct^2)} \tag{7-33}$$

where R(t) is the in-tolerance probability at time t and a, b and c are the model coefficients.

The mortality drift model is useful for parameters that are characterized by a slowly varying out-of-tolerance rate.

To implement the mortality drift model you need to know either of the following:

1. The value of the model coefficients, a, b and c.

2. The BOP, AOP, and EOP in-tolerances.

7.4.6 <u>Warranty Model</u>. The warranty reliability model is defined by the mathematical equation

$$R(t) = \frac{1}{1 + e^{a(t-b)}} \tag{7-34}$$

where R(t) is the in-tolerance probability at time t, and a and b are the model coefficients.

The warranty model is useful for parameters that tend to stay in-tolerance until reaching a well-defined cut-off time, at which point, they go out-of-tolerance.

To implement the warranty model you need to know either of the following:

1. The value of the model coefficients, a and b.

2. The BOP and EOP in-tolerance probabilities.

7.4.7 Random Walk Model. The random walk reliability model is defined by the mathematical equation

$$R(t) = erf\left(\frac{1}{\sqrt{a+bt}}\right) \tag{7-35}$$

where $R(t)$ is the in-tolerance probability at time t, and a and b are the model coefficients.

The random walk model is useful for parameters whose values fluctuate in a purely random way with respect to magnitude and direction (positive or negative).

To implement the random walk model you need to know either of the following:

1. The value of the model coefficients, a and b.

2. The BOP and EOP in-tolerance probabilities.

7.4.8 Restricted Random Walk Model. The restricted random walk reliability model is defined by the mathematical equation

$$R(t) = erf\left(\frac{1}{\sqrt{a+b\left(1-e^{-ct}\right)}}\right) \tag{7-36}$$

where $R(t)$ is the in-tolerance probability at time t, and a, b, and c are the model coefficients.

The restricted random walk model is similar to the random walk model, except that parameter fluctuations are confined within a restricted region around a mean or nominal value.

To implement the restricted random walk model you need to know either of the following:

1. The value of the model coefficients, a, b, and c.

2. The BOP, AOP, and EOP in-tolerances.

CHAPTER 8

BAYESIAN ANALYSIS
STATISTICAL MEASUREMENT PROCESS CONTROL (SMPC)

8.0 General

This chapter discusses the principles and methods used in Bayesian or SMPC analysis.[16] SMPC is a way of estimating the value of a subject parameter based on measurements made by a measuring parameter. SMPC also provides an estimate of the value of the measuring parameter, based on the same measurements. In doing so, SMPC formally recognizes that measuring parameters are not perfect.

Uncertainty analysis in general acknowledges this fact by attempting to estimate the uncertainty associated with measurements. A typical result of an uncertainty analysis is a statement of a measured value accompanied by an estimate of the uncertainty in the value. While the uncertainty estimate stands as an admission that uncertainty exists in a measurement, the measured value or a statistical mean of this value is usually taken at face value.

This apparent contradiction is especially interesting when the subject parameter is a toleranced quantity, as in the case of calibrating and testing. To see this, imagine that we arbitrarily measure a very accurate subject parameter (i.e., one with tight tolerances and a high in-tolerance probability) with a moderately accurate measuring parameter.

Obviously, no reasonable person would presume that the measured value or the mean of a sample of measured values provides a good indication of the bias in the subject parameter. Instead, one would be inclined to turn things around and use the measurement result as an indication of the bias in the measuring parameter. If we did this, we would be taking a step toward understanding SMPC.

8.1 Definitions

8.1.1 *a posteriori* value. The value calculated after taking measurements.

8.1.2 *a priori* value. The value indicated prior to taking measurements.

8.1.3 Estimated True Value. The value of a quantity obtained by SMPC (Bayesian) analysis.

8.2 SMPC Methodology

The fundamental principle that is central to SMPC analysis states that: In measurement situations where we have a priori knowledge of measuring parameter and subject parameter accuracies, the roles of measuring parameter and subject parameter are reversible. This is called the Principle of Measurement Symmetry.

A priori knowledge is the information we have about the measuring and subject parameter before any measurements are made. A posteriori estimates are estimates computed after taking measurements. As one would expect, SMPC analysis is sensitive to all measurement error

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[16] SMPC methods and concepts presented herein were developed by Dr. H. Castrup of Integrated Sciences Group (see References.

sources, not just parameter bias. Therefore, SMPC analysis is most effective when all measurement errors and their associated uncertainties are also estimated. By including measurement uncertainty, you are providing additional information from which to revise or refine the SMPC estimates.

The SMPC method derives in-tolerance probabilities and attribute biases for both a unit under test (UUT) and a set of independent test and measuring instruments (TME). The derivation of these quantities is based on measurements of a UUT attribute value made by the TME set and on certain information regarding UUT and TME attribute uncertainties. The method accommodates arbitrary accuracy ratios between TME and UUT attributes and applies to TME sets comprised of any number of instruments.

To minimize abstraction of the discussion, the treatment focuses on restricted cases in which both TME and UUT attribute values are normally distributed and are maintained within two-sided symmetric tolerance limits. This should serve to make the mathematics more concrete and more palatable. Despite these mathematical restrictions, the methodological framework is entirely general. Extension to cases involving one-sided tolerances and asymmetric attribute distributions merely calls for more mathematical brute force.

8.2.1 <u>Computation of UUT In-tolerance Probability</u>. Whether a UUT provides a stimulus, indicates a value, or shows an inherent property, the declared value of its output, indicated value, or inherent property, is said to reflect some underlying "true" value. A frequency reference is an example of a stimulus, a frequency meter reading is an example of an indicated value, and a gage block dimension is an example of an inherent property. Suppose for example that the UUT is a voltmeter measuring a (true) voltage of 10.01 mV. The UUT meter reading (10.00 mV or 9.99 mV, or some such) is the UUT's "declared" value. As another example, consider a 5 cm gage block. The declared value is 5 cm. The unknown true value (gage-block dimension) may be 5.002 cm, or 4.989 cm, or some other value.

The UUT declared value is assumed to deviate from the true value by an unknown amount. Let Y_0 represent the UUT attribute's declared value and define a random variable ε_0 as the deviation of Y_0 from the true value. The variable ε_0 is assumed a priori to be normally distributed with zero mean and variance σ_0^2. The tolerance limits for ε_0 are labeled $\pm L_0$, i.e., the UUT is considered in-tolerance if $-L_0 \leq \varepsilon_0 \leq L_0$.

A set of n independent measurements are also taken of the true value using n TME. Let Y_i be the declared value representing the ith TME's measurement. The observed differences between UUT and TME declared values are labeled according to

$$X_i \equiv Y_0 - Y_i, \quad i = 1, 2, \cdots, n \tag{8-1}$$

where the quantities X_i are assumed to be normally distributed random variables with variances σ_i^2 and mean ε_0.

Designating the tolerance limits of the ith TME attribute by $\pm L_i$, the ith TME is considered in-tolerance if $\varepsilon_0 - L_i \leq X_i \leq \varepsilon_0 + L_i$. In other words, populations of TME measurements are not expected to be systematically biased. This is the usual assumption made when TME are chosen either randomly from populations of like instruments or when no foreknowledge of TME bias is available. *Individual* unknown TME biases *are* assumed to exist. Accounting for this bias is done by treating individual instrument bias as a random variable and estimating its variance.

In applying SMPC methodology, we work with a set of variables r_i, called *dynamic accuracy*

ratios (or dynamic inverse uncertainty ratios) defined according to

$$r_i = \frac{\sigma_0}{\sigma_i}, \quad i = 1, 2, \cdots, n \tag{8-2}$$

The adjective "dynamic" will distinguish these accuracy ratios from their usual static or "nominal" counterparts, defined by L_0 / L_i, $i = 1, 2, \cdots, n$. The use of the word "dynamic" underscores the fact that each r_i defined by equation (8-2) is a quantity that changes as a function of time passed since the last calibrations of the UUT and the ith of the TME. The dynamic character exists because generally both UUT and TME population standard deviations (bias uncertainties) grow with the time since calibration.

Let P_0 be the probability that the UUT is in-tolerance at some give time since calibration. Using these definitions, we can write

$$P_0 = F(a_+) + F(a_-) - 1, \tag{8-3}$$

where $F(\cdot)$ is the distribution function for the normal distribution defined by

$$F(a_\pm) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a_\pm} e^{-\zeta^2/2} d\zeta , \tag{8-4}$$

and where

$$a_\pm = \frac{\sqrt{1 + \sum r_i^2} \left(L_0 \pm \dfrac{\sum X_i r_i^2}{1 + \sum r_i^2} \right)}{\sigma_0}. \tag{8-5}$$

In these expressions and in others to follow, all summations are taken over $i = 1, 2, \ldots, n$. The derivation of equations (8-3) and (8-5) is presented in Section 8.2.6. Note that the time dependence of P_0 is in the time dependence of a_+ and a_-. The time dependence of a_+ and a_- is, in turn, in the time dependence of r_i.

8.2.2 <u>Computation of TME In-Tolerance Probability</u>. Just as the random vari-ables $X_1, X_2, \cdots, X_n$ are TME-measured deviations from the UUT declared value, they are also UUT-measured deviations from TME declared values.

Therefore, it is easy to see that by reversing its role, the UUT can act as a TME. In other words, *any* of the n TME can be regarded as the UUT, with the original UUT performing the service of a TME. For example, focus on the ith (arbitrarily labeled) TME and swap its role with that of the UUT. This results in the following transformations:

$$X_1' = X_1 - X_i$$
$$X_2' = X_2 - X_i$$
$$\vdots$$
$$X_i' = -X_i$$
$$\vdots$$
$$X_n' = X_n - X_i,$$

where the primes indicate a redefined set of measurement results.

Using the primed quantities, the in-tolerance probability for the ith TME can be determined just as the in-tolerance probability for the UUT was determined earlier. The process begins with calculating a new set of dynamic accuracy ratios. First, we set

$$\sigma_i' = \sigma_i, \ \sigma_1' = \sigma_1, \ \sigma_2' = \sigma_2, \cdots, \sigma_i' = \sigma_0, \ \sigma_n' = \sigma_n.$$

Given these label reassignments, the needed set of accuracy ratios can be obtained using equation (8-2), i.e.,

$$r_i' = \sigma_i' / \sigma_i', \quad i = 1, 2, \cdots, n.$$

Finally, the tolerance limits are relabeled for the UUT and the ith TME according to $L_0' = L_i$ and $L_i' = L_0$.

If we designate the in-tolerance probability for the ith TME by P_i and we substitute the primed quantities obtained above, equations (8-3) and (8-5) become

$$P_i = F(a_+') + F(a_-') - 1,$$

and

$$a_\pm' = \frac{\sqrt{1 + \sum r_i'^2}\left(L_0' \pm \dfrac{\sum X_i' r_i'^2}{1 + \sum r_i'^2}\right)}{\sigma_0'}.$$

Applying similar transformations yields in-tolerance probabilities for the remaining n-1 TME.

8.2.3 <u>Variance in Parameter Bias</u>. Computing the uncertainties in UUT and TME attribute biases involves establishing the relationship between attribute uncertainty growth and time since calibration. Several models have been used to describe this relationship (see Section B.9).

To illustrate the computation of bias uncertainties, the simple negative exponential model will be used here. With the exponential model, if t represents the time since calibration, then the corresponding in-tolerance probability $R(t)$ is given by

$$R(t) = R(0)e^{-\lambda t}, \tag{8-6}$$

where the parameter λ is the out-of-tolerance rate associated with the instrument in question, and $R(0)$ is the in-tolerance probability immediately following calibration.

With the exponential model, for a given end-of-period in-tolerance target, R^*, the parameters λ and $R(0)$ determine the calibration interval for a population of instrument attributes according to

$$t = -\frac{1}{\lambda}\exp\left\{\frac{t}{T}\ln\left[\frac{R^*}{R(0)}\right]\right\}. \tag{8-7}$$

Rearranging equation (8-7) and substituting in equation (8-6) gives

$$R(t) = R(0)\exp\left\{\frac{t}{T}\ln\left[\frac{R^*}{R(0)}\right]\right\}. \tag{8-8}$$

For an instrument attribute whose acceptable values are bounded within tolerance limits ±L, the in-tolerance probability can also be written, assuming a normal distribution, as

$$R(t) = \frac{1}{\sqrt{2\pi\sigma_b^2}}\int_{-L}^{L} e^{-\zeta^2/2\sigma_b^2}\,d\zeta\ , \tag{8-9}$$

where σ_2 is the expected variance of the attribute bias at time t.

Equating equation (8-9) to equation (8-8) and rearranging yields the attribute bias standard deviation

$$\sigma_b = \frac{L}{F^{-1}\left\{\frac{1}{2}\left[1 + R(0)\exp\left(\frac{t}{T}\ln\left[\frac{R^*}{R(0)}\right]\right)\right]\right\}}, \tag{8-10}$$

where $F^{-1}(\cdot)$ is the inverse of the normal distribution function defined in equation (8-4).

Substituting L_i, T_i, t_i, $R_i(0)$, and R_i^*, $i = 0, 1, \cdots, n$, in equation (D.10) for L, T, t, $R(0)$, and R^* yields the desired instrument bias standard deviations. The variable t_i is the time passed since calibration of the UUT ($i = 0$) or of the ith TME ($i = 1, 2, \ldots, n$).

8.2.4 Accounting for Bias Fluctuations. Each attribute bias standard deviation is a component of the uncertainty in the attribute's value. Bias uncertainty represents long-term growth in uncertainty about our knowledge of attribute values. Such uncertainty growth arises from random and/or systematic processes exerted over time. Another component of uncertainty stems from such intermediate-term processes as those associated with ancillary equipment variations, environmental cycles, and diurnal electrical power level cycles.

Uncertainty contributions due to intermediate-term random variations in attribute values usually must be estimated heuristically on the grounds of engineering expectations. In the parlance of the ISO GUM, such estimates are called Type B uncertainties. Youden, for example, provides a graphical method for qualitatively evaluating contributions from human factors, laboratory processes, and reference standards. Development of a quantitative method is a subject of current research. For now, heuristic estimates are usually the best available. Heuristic estimates should represent upper bound (i.e., 3σ) one-sided limits for process uncertainty magnitudes. Experienced metrologists can often provide reasonable guesses for these limits. If we denote upper bounds for heuristically estimated contributions by δ_i, $i = 1, 2, \ldots, n$, the corresponding 3σ standard deviation is given by

$$\sigma_{\delta_i} = \delta_i/3. \tag{8-11}$$

8.2.5. Treatment of Multiple Measurements. In previous discussions, the quantities X_i are treated as single measurements of the difference between the UUT attribute and the ith TME's attribute. Yet, in most applications, testing or calibration of workload items is not limited to

8-5

single measurements. Instead, multiple measurements are usually taken. Instead of n individual measurements, we will ordinarily be dealing with n sets or *samples* of measurements.

In these samples, let n_i be the number of measurements taken using the ith TME's attribute, and let

$$X_{ij} = Y_0 - Y_{ij}$$

be the jth of these measurements. The sample mean and standard deviation are given in the usual way:

$$X_i = \frac{1}{n_i} \sum_{j=1}^{n} X_{ij} \tag{8-12}$$

and

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n} \left(X_{ij} - X_i \right)^2 . \tag{8-13}$$

The variance associated with the mean of measurements made using the ith TME's attribute is given by

$$\sigma_i^2 = \sigma_{b_i}^2 + s_i^2 / n_i + \sigma_{\delta_i}^2 ,$$

where the variables σ_{bi} and $\sigma_{\delta i}$ are the long-term and intermediate-term attribute bias standard deviations, respectively, as defined in Section 8.2.4. The square root of this variance will determine the quantities r_i defined in equation (8-2).

Note that including sample variances is restricted to the estimation of TME attribute variances. UUT attribute variance estimates contain only the terms σ_{bi} and $\sigma_{\delta i}$. This underscores what is sought in constructing the pdf $f(\varepsilon_0 | \mathbf{X})$. What we seek are estimates of the in-tolerance probability and bias of the UUT attribute. In this, we are interested in the attribute as an entity distinct from process uncertainties involved in its measurement.

It is important to keep these considerations in mind when the UUT and the ith TME switch roles. What we are after in that event is information on the attribute of the ith TME as a distinct entity. Therefore, the suitable transformations are

$$\sigma_0' = \sqrt{\sigma_{b_i}^2 + \sigma_{\delta_1}^2}$$

$$\sigma_1' = \sqrt{\sigma_{b_1}^2 + s_1^2 / n_1 + \sigma_{\delta_1}^2}$$

$$\vdots$$

$$\sigma_i' = \sqrt{\sigma_{b_i}^2 + s_i^2 / n_i + \sigma_{\delta_i}^2} \tag{8-14}$$

$$\vdots$$

$$\sigma_n' = \sqrt{\sigma_{b_n}^2 + s_n^2 / n_n + \sigma_{\delta_n}^2} .$$

Other expressions are the same as those used in treating single measurement cases. The relationship of uncertainty variables to one another is shown in Figure 8-1.
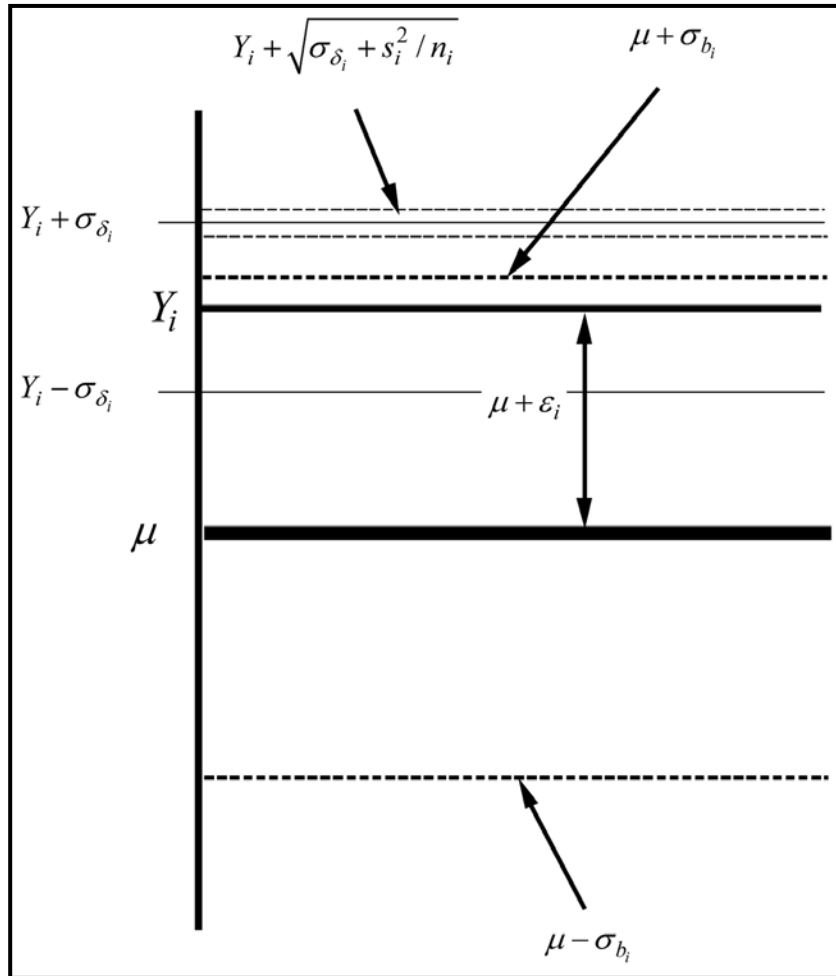
Figure 8-1. Measurement Uncertainty Components.

The standard deviation σ_{bi} provides an indication of the uncertainty in the bias of the ith instrument's attribute. The variable $\sigma_{\delta i}$ is a heuristic estimate of the standard deviation associated with intermediate-term random fluctuations in this bias. The variable s_i represents the short-term process uncertainty accompanying measurements made with the ith instrument's attribute.

8.2.6 Derivation of Equation (8-3). Let the vector X represent the random variables X1, X2, ... , Xn obtained from n independent TME measurements of ε0. We seek the conditional pdf for ε0, given X, that will, when integrated over [-L0, L0], yield the conditional probability P0 that the UUT is in-tolerance. This pdf will be represented by the function f (ε0 | X). From basic probability theory, we have

$$f(\varepsilon_0|\mathbf{X}) = \frac{f(\mathbf{X}|\varepsilon_0)f(\varepsilon_0)}{f(\mathbf{X})},$$

(8-15)

where

$$f(\varepsilon_0) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\varepsilon_0^2/2\sigma_0^2}. \tag{8-16}$$

In equation (8-15), the pdf $f(\mathbf{X}|\varepsilon_0)$ is the probability density for observing the set of measurements $X_1, X_2, \ldots, X_n$, given that the bias of the UUT is ε_0. The pdf f(ε_0) is the probability density for UUT biases.

Since the components of $\mathbf{X}$ are statistically independent, we can write

$$f(\mathbf{X}|\varepsilon_0) = f(X_1|\varepsilon_0)f(X_2|\varepsilon_0)\cdots f(X_n|\varepsilon_0), \tag{8-17}$$

where

$$f(X_i|\varepsilon_0) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(X_i-\varepsilon_0)^2/2\sigma_i^2}, \quad i=1,2,\cdots,n. \tag{8-18}$$

Note that equation (8-18) states that, for the present discussion, we assume the measurements of ε_0 to be normally distributed with a population mean value of ε_0 (the UUT "true" value) and a standard deviation σ_i. At this point, we do not provide for an unknown bias in the ith TME.[17] As we will see, the SMPC methodology will be used to estimate this bias, based on the results of measurement and on estimated measurement uncertainties.

Combining equations (8-15) through (8-18) gives

$$\begin{aligned}
f(\mathbf{X}|\varepsilon_0)f(\varepsilon_0) &= C\exp\left\{-\frac{1}{2}\left[\frac{\varepsilon_0^2}{\sigma_0^2} + \sum_{i=1}^{n}\frac{(X_i-\varepsilon_0)^2}{\sigma_i^2}\right]\right\} \\
&= C\exp\left\{-\frac{1}{2\sigma_0^2}\left[\varepsilon_0^2 + \sum_{i=1}^{n}r_i^2(X_i-\varepsilon_0)^2\right]\right\} \\
&= Ce^{-G(\mathbf{X})}\exp\left\{-\frac{1}{2\sigma_0^2}\left[\left(1+\sum r_i^2\right)\left(\varepsilon_0^2 - \frac{\sum X_i r_i^2}{1+\sum r_i^2}\right)^2\right]\right\},
\end{aligned} \tag{8-19}$$

where C is a normalization constant. The function $G(\mathbf{X})$ contains no ε_0 dependence and its explicit form is not of interest in this discussion.

The pdf $f(\mathbf{X})$ is obtained by integrating equation (8-19) over all values of ε_0. To simplify the notation, we define

$$\alpha = \sqrt{1+\Sigma r_i^2} \tag{8-20}$$

and

$$\beta = \frac{\Sigma X_i r_i^2}{1+\Sigma r_i^2}. \tag{8-21}$$

---

[17] It can be readily shown that, if the bias of a TME is unknown, the best estimate for the *population* of its measurements is the true value being measured, i.e., zero bias. This is an important *a priori* assumption in applying the SMPC methodology.

Using equations (8-20) and (8-21) in equation (8-19) and integrating over ε_0 gives

$$
\begin{aligned}
f(\mathbf{X}) &= Ce^{-G(\mathbf{X})} \int_{-\infty}^{\infty} e^{-\alpha^2(\varepsilon_0-\beta)^2/2\sigma_0^2} d\varepsilon_0 \\
&= Ce^{-G(\mathbf{X})} \frac{\sqrt{2\pi}\sigma_0}{\alpha}.
\end{aligned}
\tag{8-22}
$$

Dividing equation (8-22) into equation (8-19) and substituting in equation (8-15) yields the pdf

$$
f(\varepsilon_0|\mathbf{X}) = \frac{1}{\sqrt{2\pi}(\sigma_0/\alpha)} e^{-(\varepsilon_0-\beta)^2/2(\sigma_0/\alpha)^2}.
\tag{8-23}
$$

As we can see, ε_0 conditional on $\mathbf{X}$ is normally distributed with mean β and standard deviation σ_0/α. The in-tolerance probability for the UUT is obtained by integrating equation (8-23) over $[-L_0, L_0]$. With the aid of equation (8-5), this results in

$$
\begin{aligned}
P_0 &= \frac{1}{\sqrt{2\pi}(\sigma_0/\alpha)} \int_{-L_0}^{L_0} e^{-(\varepsilon_0-\beta)^2/2(\sigma_0/\alpha)^2} d\varepsilon_0 \\
&= \frac{1}{\sqrt{2\pi}} \int_{-(L_0-\beta)/(\sigma_0/\alpha)}^{(L_0-\beta)/(\sigma_0/\alpha)} e^{-\zeta^2/2} d\zeta \\
&= F(a_-) - F(a_+) \\
&= F(a_+) + F(a_-) - 1,
\end{aligned}
$$

which is equation (8-3) with α and β as defined in equations (8-20) and (8-21).

8.2.7 <u>Estimation of Biases</u>. Obtaining the conditional pdf $f(\varepsilon_0|\mathbf{X})$ allows us to compute moments of the UUT attribute distribution. Of particular interest is the first moment, or *distribution mean*. The UUT distribution mean is the conditional expectation value for the bias ε_0. Thus, the UUT attribute bias is estimated by

$$
\begin{aligned}
\beta_0 &= E(\varepsilon_0|\mathbf{X}) \\
&= \int_{-\infty}^{\infty} \varepsilon_0 f(\varepsilon_0|\mathbf{X}) d\varepsilon_0.
\end{aligned}
\tag{8-24}
$$

Substituting from equation (8-23) and using equation (8-21) gives

$$
\beta_0 = \frac{\Sigma X_i r_i^2}{1 + \Sigma r_i^2}.
\tag{8-25}
$$

Similarly, bias estimates can be obtained for the TME set by making the transformations described in Section 8.2.2; for example, the bias of TME 1 is given by

$$\beta_1 = E(\varepsilon_1 | \mathbf{X}) = \frac{\Sigma X_i' r_i'^2}{1 + \Sigma r_i'^2} . \tag{8-26}$$

To exemplify bias estimation, let use consider a proficiency audit question in which three different pressure measuring instruments are used to measure a 10,000 psi nominal pressure source. All three instruments have a specified tolerance of ± 10 psi. Instrument 1 reads 0 psi difference from nominal ($Y_0 = 0$), instrument 2 reads 6 psi higher than nominal ($Y_1 = 6$), and instrument 3 reads 15 psi higher than nominal ($Y_2 = 15$).

In this example, instrument 1 is designated as the UUT, instrument 2 as TME 1 and instrument 3 as TME 2. For simplicity, we set $R(0) = 1$, and bias fluctuation and process uncertainties equal to zero. Thus,

$$X_1 = Y_0 - Y_1$$
$$= -6$$
$$X_2 = Y_0 - Y_2$$
$$= -15,$$

and

$$r_1 = r_2 = 1.$$

Unless otherwise shown, we can assume that the in-tolerance probabilities for all three instruments are about equal to their average-over-period values.

For the instruments used in the proficiency audit, it was determined that the population uncertainty is managed to achieve an in-tolerance probability of $R^* = 0.72$ at the end of the calibration interval. We assume that we can use average-over-period in-tolerance probabilities for $R(t)$ in this example.

With the exponential model, if $R(0) = 1$, the average in-tolerance probability is roughly equal to the in-tolerance probability halfway through the calibration interval. Thus, setting $t = T/2$ in Equation (8-10) yields

$$\sigma_0 = \frac{10}{F^{-1}\left\{\frac{1}{2}\left[1 + \exp\left(\frac{1}{2}\ln 0.72\right)\right]\right\}}$$
$$= \frac{10}{F^{-1}(0.92)}$$
$$= 10/1.43$$
$$= 6.97.$$

Substituting in the expression for $a_\pm$ above gives

$$a_\pm = \frac{\sqrt{3}(10 \mp 7)}{6.97}$$
$$= 2.49 \pm 1.74.$$

Thus, the in-tolerance probability for the UUT (instrument 1) is

$$P_0 = F(0.75) + F(4.23) - 1$$
$$= 0.77 + 1.00 - 1$$
$$= 0.77.$$

To compute the in-tolerance probability for TME 1 (instrument 2), the UUT and TME 1 swap roles. By using the transformations of Section 8.2.2, we have

$$X_1^{'} = -X_1$$
$$= 6$$
$$X_2^{'} = X_2 - X_1$$
$$= -9$$

in place of X_1 and X_2 in Equation (8-5). Recalling that $\sigma_0^{'} = \sigma_0$ in this example gives

$$a_\pm^{'} = \frac{\sqrt{1 + (1+1)}\left[10 \pm \dfrac{6 - 9}{1 + (1+1)}\right]}{\sigma_0^{'}}$$
$$= \frac{\sqrt{3}(10 \pm 1)}{6.97}$$
$$= 2.49 \pm 0.25.$$

Thus, by Equation (8-3), the in-tolerance probability for TME 1 (instrument 2) is

$$P_1 = F(2.24) + F(2.73) - 1$$
$$= 0.99 + 1.00 - 1$$
$$= 0.99.$$

In computing the in-tolerance probability for TME 2, the UUT and TME 2 swap roles. Thus

$$X_1^{'} = X_1 - X_2$$
$$= 15$$
$$X_2^{'} = -X_2$$
$$= 9.$$

Using these quantities in Equation (8-5) and setting $\sigma_0^{'} = \sigma_0$ gives

$$a_\pm^{'} = 2.49 \pm 1.99.$$

Thus, by Equation (8-3), the in-tolerance probability for TME 2 (instrument 3) is

$$P_2 = F(4.47) + F(0.50) - 1$$
$$= 1.00 + 0.69 - 1$$
$$= 0.69.$$

By summarizing these results, we estimate a roughly 77% in-tolerance probability for instrument 1, a 99% in-tolerance probability for instrument 2, and a 69% in-tolerance probability for instrument 3.

As previously stated, all three instruments are managed to the same R^* target, have the same tolerances, and are calibrated in the same way using the same equipment and procedures. Therefore, their standard deviations when the measurements were made should be about equal.

By using equations (8-25) and (8-26) and by recalling that $\sigma_0 = \sigma_1 = \sigma_2$, we get

$$\text{Instrument 1 (UUT) bias: } \beta_0 = \frac{-6-15}{1+(1+1)} = -7$$

$$\text{Instrument 2 (TME 1) bias: } \beta_1 = \frac{6-9}{1+(1+1)} = -1$$

$$\text{Instrument 3 (TME 2) bias: } \beta_2 = \frac{15+9}{1+(1+1)} = 8.$$

If desired, these bias estimates could serve as correction factors for the three instruments. If used in this way, the quantity 7 would be added to all measurements made with instrument 1. The quantity 1 would be added to all measurements made with instrument 2. And, the quantity 8 would be subtracted from all measurements made with instrument 3.[18]

Note that all biases are within the stated tolerance limits (±10) of the instruments, which might encourage users to continue to operate their instruments with confidence. However, the computed in-tolerance probabilities showed only a 77% chance that instrument 1 was in-tolerance and an even lower 69% chance that instrument 3 was in-tolerance. Such results tend provide valuable information from which to make cogent judgments regarding instrument disposition.

8.2.8 <u>Bias Confidence Limits</u>. Another variable that can be useful in making decisions based on measurement results is the range of the confidence limits for the estimated biases. Estimating confidence limits for the computed biases β_0 and β_i, $i = 1, 2, \cdots, n$, means first determining the statistical probability density functions for these biases. From equation (8-25) we can write

---

[18] Since all three instruments are considered *a priori* to be of equal accuracy, the best estimate of the true value of the measurand would be the average of the three measured deviations: $\varepsilon_0 = (0 + 6 + 15)/3 = 7$. Thus, a zero reading would be indicative of a bias of -7, a +6 reading would be indicative of a bias of -1, and a +15 reading would be indicative of a bias of +8. These are the same estimates we obtained with SMPC. Obviously, this is a trivial example. This become more interesting when each measurement has a different uncertainty, i.e., when $\sigma_0 \neq \sigma_1 \neq \sigma_2$.

$$\beta_0 = \sum_{i=1}^{n} c_i X_i , \tag{8-27}$$

where

$$c_i = \frac{r_i^2}{1 + \Sigma r_i^2} . \tag{8-28}$$

With this convention, the probability density function of β_0 can be written:

$$\begin{aligned} f(\beta_0) &= f(\Sigma c_i X_i) \\ &= f(\Sigma \Psi_i), \end{aligned} \tag{8-29}$$

where

$$\Psi_i = c_i X_i . \tag{8-30}$$

Although the coefficients c_i, $i = 1, 2, \ldots , n$, are in the strictest sense random variables, to a first approximation, they can be considered fixed coefficients of the variables X_i. Since these variables are normally distributed (see equation (8-18)), the variables ψ_i are also normally distributed. The appropriate expression is

$$f(\Psi_i) = \frac{1}{\sqrt{2\pi}\sigma_{\Psi_i}} e^{-(\Psi_i - \eta_i)/2\sigma_{\Psi_i}^2} , \tag{8-31}$$

where

$$\sigma_{\Psi_i} = c_i \sigma_i \tag{8-32}$$

and

$$\eta_i = c_i \varepsilon_0 . \tag{8-33}$$

Since the variables ψ_i are normally distributed, their linear sum is also normally distributed:

$$\begin{aligned} f(\Sigma \psi_i) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(\Sigma \Psi_i - \eta)/2\sigma^2} \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(\beta_0 - \eta)^2/2\sigma^2} \\ &= f(\beta_0), \end{aligned} \tag{8-34}$$

where

$$\sigma = \sqrt{\Sigma \sigma_{\psi_i}^2} , \tag{8-35}$$

and

$$\eta = \Sigma \eta_i . \tag{8-36}$$

Equation (8-34) can be used to find the upper and lower confidence limits for β_0. Denoting these limitsd by β_0^+ and β_0^-, if the desired level of confidence is p x 100% then

$$p = \int_{\beta_0^-}^{\beta_0^+} f(\mathrm{B}_0)d\mathrm{B}_0 ,$$

or

$$\int_{\beta_0^-}^{\infty} f(\mathrm{B}_0)d\mathrm{B}_0 = (1-p)/2 = \int_{\infty}^{\beta_0^+} f(\mathrm{B}_0)d\mathrm{B}_0 .$$

Integrating equation (8-34) from β_0^+ to ∞ and using equations (8-35) and (8-36) yields

$$1 - F\left(\frac{\beta_0^+ - \eta}{\sigma}\right) = (1-p)/2$$

and

$$F\left(\frac{\beta_0^+ - \eta}{\sigma}\right) = (1+p)/2 .$$

Solving for β_0^+ gives

$$\beta_0^+ = \eta + \sigma F^{-1}\left(\frac{1+p}{2}\right). \tag{8-37}$$

Solving for the lower confidence for β_0^- in the same manner, we begin with

$$\int_{-\infty}^{\beta_0^-} f(\beta_0)d\beta_0 = (1-p)/2 .$$

This yields, with the aid of equation (8-24),

$$F\left(\frac{\beta_0^- - \eta}{\sigma}\right) = (1-p)/2 . \tag{8-38}$$

Using the following property of the normal distribution

$$F(-x) = 1 - F(x),$$

we can rewrite equation (8-38) as

$$F\left(-\frac{\beta_0^- - \eta}{\sigma}\right) = 1 - (1-p)/2$$

$$= (1+p)/2,$$

where

$$\beta_0^- = \eta - \sigma F^{-1}\left(\frac{1+p}{2}\right). \tag{8-39}$$

From equation (8-34), the parameter η is seen to be the expectation value for β_0. Our best available estimate for this quantity is the computed UUT bias, namely β_0 itself. We thus write the computed upper and lower confidence limits for β_0 as

$$\beta_0^\pm = \beta_0 \pm \sigma F^{-1}\left(\frac{1+p}{2}\right). \tag{8-40}$$

In like fashion, we can write down the solutions for the TME biases β_i, $i = 1, 2, \cdots, n$:

$$\beta_i^\pm = \beta_i \pm \sigma' F^{-1}\left(\frac{1+p}{2}\right), \tag{8-41}$$

where

$$\sigma' = \sqrt{\Sigma c_i'^2 \sigma_i'^2}, \tag{8-42}$$

and

$$c_i' = \frac{r_i'^2}{1 + \Sigma r_j'^2}. \tag{8-43}$$

The variables r_i' in this expression are defined as before.

To illustrate the determination of bias confidence limits, we again turn to the proficiency audit example. In this example where

$$\sigma_0 = \sigma_1 = \sigma_2 = 6.97,$$

and

$$r_1 = r_2 = r_3 = 1.$$

By equations (8-28) and (8-34),

$$c_i = c_i' = \frac{1}{3},$$

and

$$\sigma = \sqrt{\frac{\sigma_1^2}{9} + \frac{\sigma_2^2}{9}}$$
$$= \frac{\sqrt{2}\sigma_0}{3}$$
$$= 3.29$$
$$= \sigma'.$$

Substituting in equations (8-40) and (8-41) yields

$$\beta_0^\pm = \beta_0 \pm 3.29 F^{-1}\left(\frac{1+p}{2}\right),$$

$$\beta_1^\pm = \beta_1 \pm 3.29 F^{-1}\left(\frac{1+p}{2}\right),$$

and

$$\beta_2^\pm = \beta_2 \pm 3.29 F^{-1}\left(\frac{1+p}{2}\right).$$

Suppose that the desired confidence level is 95%. Then p = 0.95, and

$$F^{-1}\left(\frac{1+p}{2}\right) = F^{-1}(0.975)$$
$$= 1.96,$$

and

$$3.29 F^{-1}\left(\frac{1+p}{2}\right) = 6.4.$$

Since $\beta_0 = -7$, $\beta_1 = -1$, and $\beta_2 = +8$, this result, when substituted in the above expressions, gives 95% confidence limits for the estimated biases:

$$-13.4 \le \beta_0 \le -0.6$$
$$-7.4 \le \beta_1 \le 5.4$$
$$1.6 \le \beta_2 \le 14.4.$$

8.3 Application of SMPC

As with any analysis technique, the results are only as good as the information or data input into it. Or as the saying goes: "garbage in, garbage out." To avoid misusing SMPC, there are two provisions to keep in mind.

1. At its present stage of development, SMPC is strictly applicable when process errors and measurement biases are normally distributed. For process errors or parameter bias with a containment probability (confidence level) less than 100%, UncertaintyAnalyzer assumes a normal distribution.

2. In using SMPC with more than one measurement sample, the samples must be based on measurements of the same quantity. SMPC needs to "anchor" measurements to a common reference for comparison and evaluation.

CHAPTER 9

SOFTWARE VALIDATION

9.0 General

This chapter discusses the protocols that Integrated Sciences Group (ISG) has developed and implemented to validate the UncertaintyAnalyzer program. Examples are presented that compare UncertaintyAnalyzer calculations to values obtained by hand calculations and from Excel spreadsheets.

9.1 Definitions

9.1.1 <u>Protocol</u>. A procedure or practice, set of rules, or code of behavior.

9.1.2 <u>Validate</u>. To confirm or prove to be valid or correct.

9.1.3 <u>Validation</u>. See Verification.

9.1.4 <u>Verification</u>. Establishment or confirmation of the truth or accuracy of a fact, theory, etc.

9.1.5 <u>Verify</u>. To make certain, to check or test the accuracy or correctness of, as by investigation, comparison with a standard, or reference to the facts.

9.2 Software Validation Protocol

Currently, there are no standards or guidelines for testing and validating uncertainty analysis software. However, there are many common-sense protocols that Integrated Sciences Group has adopted for the validation and verification of UncertaintyAnalyzer. These protocols are described below.

- Validation of mathematical and statistical methods.
- Verification of numerical approximations and calculations.
- Verification of program functionality.

9.2.1 Mathematical and Statistical Methods. In general, it is not sufficient to simply state that an uncertainty analysis application incorporates internationally accepted methods, such as those described in the ISO GUM. The software developer should also publish papers and articles that clearly describe the mathematical and statistical concepts that are incorporated in the product. This serves two purposes:

11. It shows whether or not the developer has a sufficient technical understanding of uncertainty analysis concepts and principles.

12. The information can be reviewed and scrutinized in the public domain.

Since 1992, ISG has published and presented papers at technical conferences and symposiums that specifically discuss the principles and methods of estimating measurement uncertainty. These papers are available in conference proceedings and via download from ISG's website. ISG also actively documents new uncertainty analysis methods and practices as they are developed. These articles are available for download from ISG's website for peer review.

9.2.2 <u>Numerical Approximations and Calculations</u>. Depending upon the sophistication of the numerical algorithms, program calculations can be verified via hand calculations, Excel spreadsheets, or math and statistics applications such as MathCAD or Mathematica. Verification of numerical algorithms can be achieved in a number of ways including:

- Alpha testing via internal peer review and verification.

- Beta testing via external review and verification by selected customer base.

- Widespread peer review and verification via distribution of freeware subprograms and applets.

- Large-scale customer use and feedback. This applies to well established analysis programs that have been used by 100+ customers for the past several years or more.

Since initial product release in 1994, ISG has conducted extensive internal review and verification of UncertaintyAnalyzer's subroutines and subprograms. Internal review and verification is conducted to all new program features and functions as they are added. ISG also periodically releases key features as freeware applications for external review prior to final implementation into UncertaintyAnalyzer.

9.2.3 <u>Program Functionality</u>. Another important aspect of software validation is the verification that the program screens, templates, or worksheets function as intended. For example, data entered into a drill-down screen should be properly stored and transferred to other screens as needed. The protocol for testing and validating program functionality is the same as described for numerical algorithms.

Of course, software programs cannot fully eliminate user input error. However, UncertaintyAnalyzer does contain error trapping subroutines to ensure that realistic information and data are entered in the appropriate fields and cells. UncertaintyAnayzer also contains a comprehensive Help file that is easily accessed from all screens, templates or worksheets. The Help topics are written in a concise manner that clearly conveys the appropriate information.

9.3 Validation Examples

Three uncertainty analyses are included herein to illustrate how UncertaintyAnalyzer calculations are verified and compared to values obtained from hand calculations (where possible) and from calculations performed via Excel spreadsheet.

9.3.1 <u>Cylinder Volume Measurement</u>. In this example, the cylinder is a steel artifact with nominal design dimensions of 0.65 cm length by 1.40 cm diameter. The length and diameter are measured with a micrometer. The objective is the estimate the uncertainty in the cylinder volume measurement.

The mathematical relationship between the cylinder volume in terms of length and diameter is given as

$$V = \pi L \left(\frac{D}{2} \right)^2$$

where L and D are the length and diameter components, respectively.

A comparison of the results from hand calculations and those obtained using UncertaintyAnalyzer are presented herein. Details of the uncertainty analysis are in Appendix A.

Table 9-1. Comparison of Total Uncertainty and Degrees of Freedom

	Hand Calculation	UncertaintyAnalyzer
Cylinder Volume	1.108 cm^3	1.1080 cc
Std Uncertainty	0.0194 cm^3	0.0194 cc
Deg of Freedom	170	169

The standard uncertainties and component uncertainties for the measurement process errors obtained from hand calculations are summarized in Table 9-2. Component uncertainty is obtained by multiplying the standard uncertainty by the appropriate sensitivity coefficient. The UncertaintyAnalyzer multivariate analysis report is shown in Figure 9-1.

Table 9-2. Uncertainty Estimates for Cylinder Volume using Hand Calculations

Variable Name	Standard Uncertainty	% Confidence	± Error Limits	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
L0						0.65 cm
Lbias	0.0045 cm	97.5	0.1 mm	1.613	0.00726 cm^3	0 cm
Lran	0.0029 cm			1.613	0.00467 cm^3	0.037 cm
Lres	0.0029 cm	100.0	0.05 mm	1.613	0.00467 cm^3	0 cm
Lop	0.0030 cm	90.0	0.05 mm	1.613	0.00484 cm^3	0 cm
Lenv	0.00000068 cm			1.613	0.00000110 cm^3	0 cm
D0						1.40 cm
Dbias	0.0045 cm	97.5	0.1 mm	1.546	0.00696 cm^3	0 cm
Dran	0.0042 cm				0.00649 cm^3	0.033 cm
Dres	0.0029 cm	100.0	0.05 mm		0.00448 cm^3	0 cm
Dop	0.0030 cm	90.0	0.05 mm		0.00464 cm^3	0 cm
Denv	0.00000146 cm				0.00000226 cm^3	0 cm

where

L0 = nominal cylinder length

Lbias = measurement bias in length measurement

Lran = length repeatability error

Lres = length resolution error

Lop = length operator bias

Lenv = length environmental factors error

D0 = nominal cylinder diameter

Dbias = measurement bias in diameter measurement

Dran = diameter repeatability error

Dres = diameter resolution error

Dop = diameter operator bias

Denv = diameter environmental factors error

Root Variables Analysis Results:

Variable Name	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg. Freedom	Sensitivity Coefficient	Component Uncertainty	Adjusted Mean
L0						1.6125		0.65 cm
Lbias	0.1 mm	97.50	0.045 mm	B	inf	1.6125	0.00726 cc	0 cm
Lran			0.029 mm	A	6	1.6125	0.00468 cc	0.0371 cm
Lres	0.05 mm	100.00	0.029 mm	B	inf	1.6125	0.00468 cc	0 cm
Lop	0.05 mm	90.00	0.030 mm	B	inf	1.6125	0.00484 cc	0 cm
Lenv	0.0000133 mm	95.00	0.0000068 mm	B	inf	1.6125	0.000001097 cc	-0.00000933 cm
D0						1.5465		1.40 cm
Dbias	0.1 mm	97.50	0.045 mm	B	inf	1.5465	0.00696 cc	0 cm
Dran			0.042 mm	A	6	1.5465	0.0065 cc	0.0329 cm
Dres	0.05 mm	100.00	0.029 mm	B	inf	1.5465	0.004485 cc	0 cm
Dop	0.05 mm	90.00	0.030 mm	B	inf	1.5465	0.00464 cc	0 cm
Denv	0.0000287 mm	95.00	0.0000146 mm	B	inf	1.5465	0.00000226 cc	-0.0000201 cm

Analysis Summary:
Computed Parameter Value: 1.1080 cc
Estimated Uncertainty: 0.0194 cc
Distribution: Student's t
Degrees of Freedom: 169
Analysis Category: Type A,B

Figure 9-1. Multivariate Analysis Report for Cylinder Volume Example.

9.3.2 Load Cell Calibration. In this example, a load cell (i.e., tension transducer) is calibrated using a weight standard, as illustrated in Figure 9-2. The calibration weight is extended from the load cell via a monofilament line. Repeat measurements of DC voltage are obtained by adding and removing the calibration weight. The DC voltage output from an amplifier/signal conditioner is measured with a digital multimeter.
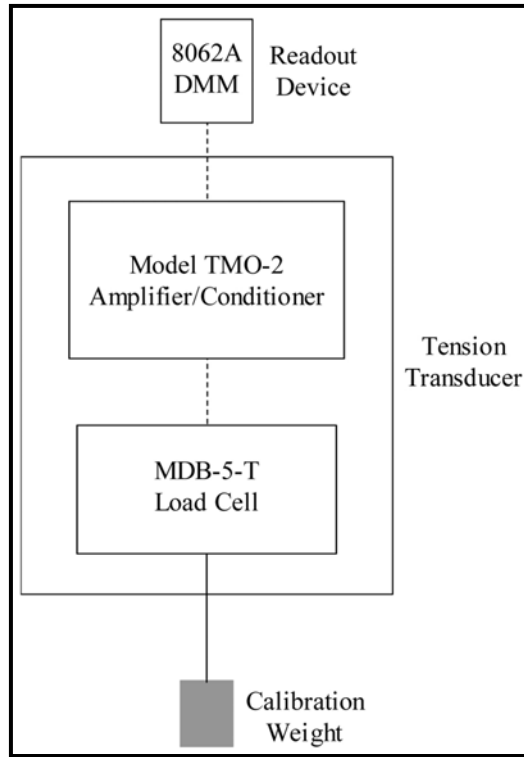
Figure 9-2. Load Cell Calibration Setup.

The uncertainty in the load cell output voltage is estimated using both a system model analysis approach and a more general multivariate analysis method. Results obtained from Excel spreadsheet analyses are compared to similar analyses using UncertaintyAnalyzer. Analysis details are presented in Appendix B.

9.3.2.1 <u>System Model Results</u>. Comparisons of the computed Load Cell Module output, total uncertainty and degrees of freedom obtained from Excel spreadsheet and Uncertainty-Analyzer are listed in Table 9-3.

Table 9-3. Comparison of Load Cell Total Uncertainty and Degrees of Freedom		
	Excel Spreadsheet	**UncertaintyAnalyzer**
Load Cell Output	8.88 mV	8.88 mV
Std Uncertainty	0.005 mV	0.00 mV
Deg. Of Freedom	Infinite	Infinite

The estimated uncertainties for the Load Cell Module obtained from Excel spreadsheet calculations are listed in Table 9-4. The UncertaintyAnalyzer output report for the Load Cell Module is shown in Figure 9-3.

Table 9-4. Spreadsheet Analysis Results for Load Cell Module						
Parameter Name	Standard Uncertainty	% Confidence	± Error Limits	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Cal_Weight	0.0388 g	99	0.1	0.00882		1,000 g
Sensitivity		95		1,000		0.00882 mV/g
Nonlinearity	0.005 mV	95.45	0.01 mV	1	0.005 mV	0 mV
Hysteresis	0.005 mV	95.45	0.01 mV	1	0.005 mV	0 mV
Noise	0.005 mV	95.45	0.01 mV	1	0.005 mV	0 mV
Random	0.0015 mV			1	0.0015 mV	0 mV
Zero_Offset	0.1 mV	95.45	0.2 mV	1	0.100 mV	0 mV
Temp_Effect_Out	0.00022 mV/ F	95.45	0.00044 mV/ F	0	0.00 mV	0 mV/deg F
Temp_Effect_Zero	0.0005 mV/ F	95.45	0.001 mV/ F	0	0.00 mV	0 mV/deg F
Temp_Range	0.776 F	99	2 F	0		10 deg F

where

Cal_Weight	= Calibration Weight
Sensitivity	= Load Cell Sensitivity
Nonlinearity	= Load Cell Nonlinearity
Hysteresis	= Load Cell Hysteresis
Noise	= Noise
Random	= Error due to Repeat Measurements
Zero_Offset	= Zero Balance
Temp_Effect_Out	= Temperature Effect on Output
Temp_Effect_Zero	= Temperature Effect on Zero
Temp_Range	= Temperature Range

Uncertainty Analysis Report   **System Module Analysis Results**

UncertaintyAnalyzer System Model Analysis
Load Cell Calibration System

14-Jul-2005

Module 1: Load Cell
Input Measurement Area: Mass
Output Measurement Area: DC Voltage

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Load Cell Sensivity	0.00882		95.00				1,000.0	
Load Cell Nonlinearity	0	0.0100	95.45	0.005	B	inf	1.0	0.005
Load Cell Hysteresis	0	0.0100	95.45	0.005	B	inf	1.0	0.005
Error due to Repeat Measurements	0.0590	0.006523	95.00	0.001528	A	2	1.0	0.001528
Noise	0	0.0100	95.45	0.005	B	inf	1.0	0.005
Zero Balance	0	0.2000	95.45	0.1	B	inf	1.0	0.1
Temperature Effect on Output	0	0.00044	95.45	0.00022	B	inf	10.0	0.00220
Temperture Effect on Zero	0	0.0010	95.45	0.0005	B	inf	10.0	0.0050
Temperature Range	10	2.0000	99.00	0.776		inf		

Module Analysis Summary:
Module Input: 1,000 g
Module Output: 8.88 mV
Output Uncertainty: 0.1005 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type A,B

Figure 9-3. Load Cell Module Report.

Comparisons of the computed Amplifier Module output, total uncertainty and degrees of freedom obtained from Excel spreadsheet and UncertaintyAnalyzer are listed in Table 9-5.

Table 9-5. Comparison of Amplifier Total Uncertainty and Degrees of Freedom		
	Excel Spreadsheet	**UncertaintyAnalyzer**
Amplifier Output	4.44 V	4.44 V
Std Uncertainty	51.6 mV	51.6 mV
Deg. Of Freedom	Infinite	Infinite

Comparison of the estimated uncertainties for the Amplifier Module obtained from Excel spreadsheet calculations are listed in Table 9-6. The Uncertainty Analyzer output report for the Amplifier Module is shown in Figure 9-4.

Table 9-6. Spreadsheet Analysis Results for Amplifier Module						
Parameter Name	Standard Uncertainty	% Confidence	± Error Limits	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Load_Cell_Output	0.1005 mV					8.88 mV
Gain		95		8.88		0.5 V/mV
Gain_Acc	2.5 mV	95.45	5 mV	1	0.0025 V	0 V
Stability	0.5 mV	95.45	1 mV	1	0.0005 V	0 V
Nonlinearity	0.5 mV	95.45	1 mV	1	0.005 V	0 V
Noise	1.165 mV	99	3 mV	1	0.001165 V	0 V
Bal_Stability	10 mV	95.45	20 mV	1	0.01 V	0 V
Temp_Coeff	1 mV/C	95.45	2 mV/C	5.6	0.0056 V	0 mV/C
Temp_Range	0.427 C	99	1.1 C		0 V	5.6 C

    where

        Load_Cell_Output   = Output from Load Cell Module
        Gain   = Amplifier Gain
        Gain_Acc   = Amplifier Accuracy
        Stability   = Amplifier Stability
        Nonlinearity   = Amplifier Nonlinearity
        Noise   = Amplifier Noise
        Bal_Stability   = Balance Stability
        Temp_Coeff   = Temperature Coefficient
        Temp_Range   = Temperature Range

```
Uncertainty Analysis Report                                    System Module Analysis Results

UncertaintyAnalyzer System Model Analysis
Load Cell Calibration System

14-Jul-2005

Module 2:  Amplifier
Input Parameter:  Load Cell
Input Measurement Area:  DC Voltage
Output Measurement Area:  DC Voltage

Analysis Results:
                              ± Error   % Con-    Standard    Analysis   Deg      Sensitivity   Component
Name                  Value   Limits    fidence   Uncertainty  Type     Freedom   Coefficient   Uncertainty
Amplifier Gain        0.5               95.00                                     8.88
Amplifier Accuracy    0       0.0050    95.45     0.0025       B        inf       1.0           0.0025
Amplifier Stability   0       0.0010    95.45     0.0005       B        inf       1.0           0.0005
Amplifier Nonlinearity 0      0.0010    95.45     0.0005       B        inf       1.0           0.0005
Amplifier Noise       0       0.003000  99.00     0.001165     B        inf       1.0           0.001165
Balance Stability     0       0.0200    95.45     0.0100       B        inf       1.0           0.0100
Temperature Coefficient 0     0.0020    95.45     0.0010       B        inf       5.60          0.0056
Temperature Range     5.6     1.1000    99.00     0.427                 inf

Module Analysis Summary:
Module Input:  8.88 mV
Input Uncertainty:  0.1005 mV
Degrees of Freedom:  Infinite
Module Output:  4.44 V
Output Uncertainty:  51.6 mV
Distribution:  Normal
Degrees of Freedom:  Infinite
Analysis Category:  Type B
```

Figure 9-4. Amplifier Module Report.

Comparisons of the computed Digital Multimeter Module output, total uncertainty and degrees of freedom obtained from Excel spreadsheet and UncertaintyAnalyzer are listed in Table 9-7. These values also represent to overall system output, total system uncertainty, and degrees of freedom.

Table 9-7. Comparison of Digital Multimeter Total Uncertainty and Degrees of Freedom		
	Excel Spreadsheet	**UncertaintyAnalyzer**
DMM Output	4.44 V	4.44 V
Std Uncertainty	51.9 mV	51.9 mV
Deg. Of Freedom	Infinite	Infinite

The estimated uncertainties for the Digital Multimeter Module obtained from Excel spreadsheet calculations are listed in Table 9-8. The UncertaintyAnalyzer output report for the Digital Multimeter Module is shown in Figure 9-5. The UncertaintyAnalyzer overall system analysis report is shown in Figure 9-6.

Table 9-8. Spreadsheet Analysis Results for Digital Multimeter Module

Parameter Name	Standard Uncertainty	% Confidence	± Error Limits	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Amplifier_Output	51.6 mV			1	51.6 mV	4.4 V
DMM_Accuracy	5.2 mV	95.45	10.47 mV	1	5.2 mV	0 V
DMM_Resolution	0.2898 mV	100	0.5 mV	1	0.289 mV	0 V

where

    Amplifier_Output   =   Output from Amplifier Module

    DMM_Accuracy   =   Digital Multimeter Accuracy

    DMM_Resolution   =   Digital Multimeter Resolution

Uncertainty Analysis Report   System Module Analysis Results

UncertaintyAnalyzer System Model Analysis
Load Cell Calibration System

14-Jul-2005

Module 3: Digital Multimeter
Input Parameter: Amplifier
Input Measurement Area: DC Voltage
Output Measurement Area: DC Voltage

Analysis Results:

Name	Value	± Error Limits	% Confidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Digital Multimeter Accuracy	0	0.01047	95.45	0.00523	B	inf	1.0	0.00523
Digitial Multimeter Resolution	0	0.0005	100.00	0.0002887	B	inf	1.0	0.0002887

Module Analysis Summary:
Module Input: 4.44 V
Input Uncertainty: 51.6 mV
Degrees of Freedom: Infinite
Module Output: 4.44 V
Output Uncertainty: 51.9 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure 9-5. Digital Multimeter Module Report.

```
Uncertainty Analysis Report                                    System Analysis Results

UncertaintyAnalyzer System Model Analysis
14-Jul-2005

System Name:  Load Cell Calibration System

Module Name                Input        Output       Uncertainty    Coefficient
Load Cell                  1,000 g      8.88 mV      0.1005 mV      0
Amplifier                  8.88 mV      4.44 V       51.6 mV        0
Digital Multimeter         4.44 V       4.44 V       51.9 mV        1.000

System Analysis Summary:
System Input:  1,000 g
System Output:  4.44 V
System Uncertainty:  51.9 mV
Confidence Level:  95.00 %
Coverage Factor:  1.9653 mV
Degrees of Freedom:  Infinite
Tolerance Limits:  102 mV
```

Figure 9-6. Load Cell Calibration System Report

9.3.2.2 <u>Multivariate Analysis Results</u>. A multivariate analysis was also conducted, in which an overall equation was entered for the load cell calibration system, along with nested variables equations. The analysis details are presented in Appendix B.

Comparisons of the total system output, uncertainty, and degrees of freedom for both analyses are listed in Table 9-9.

Table 9-9. Comparison of System Output Uncertainty and Degrees of Freedom

	Excel Spreadsheet	UncertaintyAnalyzer
System Output	4.44 V	4.44 V
Std Uncertainty	51.9 mV	51.9 mV
Deg. Of Freedom	Infinite	Infinite

The estimated uncertainties for the Load Cell Calibration System obtained from Excel spreadsheet calculations are listed in Table 9-10. The UncertaintyAnalyzer output report is shown in Figure 9-7.

Table 9-10 Spreadsheet Multivariate Analysis Results for Load Cell System						
Variable Name	**Standard Uncertainty**	**% Confidence**	**± Error Limits**	**Sensitivity Coefficient**	**Component Uncertainty**	**Nominal or Mean Value**
Cal_Weight	0.0388 g	99	0.1 g	0.00441	0.000171 V	1,000 g
Load_Cell_Nonlinearity	0.005 mV	95.45	0.01 mV	0.5	0.0025 V	0 mV
Sensitivity		95		500		0.00882 mV/g
Random	0.0015 mV			0.5	0.000764 V	0.0590 mV
Load_Cell_Hysteresis	0.005 mV	95.45	0.01 mV	0.5	0.0025 V	0 mV
Load_Cell_Noise	0.005 mV	95.45	0.01 mV	0.5	0.0025 V	0 mV
Zero_Offset	0.1 mV	95.45	0.2 mV	0.5	0.05 V	0 mV
Temp_Effect_Output	0.00022 mV/F	95.45	0.00044 mV/F	5	.0011 V	0 mV/deg F
Temp_Effect_Zero	0.0005 mV/F	95.45	0.001 mV/F	5	.0025 V	0 mV/dig F
Temp_Range	0.776 F	99	2 F			10 F
Amplifier_Gain		95		8.879		0.5 V/mV
Gain_Acc	2.5 mV	95.45	5 mV	1	0.0025 V	0 V
Amplifier_Stability	0.5 mV	95.45	1 mV	1	0.0005 V	0 V
Amplifier_Nonlinearity	0.5 mV	95.45	1 mV	1	0.0005 V	0 V
Amplifier_Noise	1.165 mV	99	3 mV	1	0.001165 V	0 V
Bal_Stability	10 mV	95.45	20 mV	1	0.01 V	0 V
Temp_Coeff	1 mV/C	95.45	2 mV/C	5.6	0.0056 V	0 V/deg C
Temp_Range_DecC	0.388 C	99	1 C			5.6 deg C
DMM_Accuracy	5.23 mV	95.45	10.47 mV	1	0.00523 V	0 V
DMM_Resolution	0.29 mV	100	0.5 mV	1	0.00029 V	0 V

UncertaintyAnalyzer Load Cell Calibration System Analysis
14-Jul-2005

Subject Parameter Data:
Parameter Name: Voltage Output from Load Cell Calibration System
Qualifier 1:
Qualifier 2:

Root Variables Analysis Results:

Variable Name	± Error Limits	% Con- fidence	Standard Uncertainty	Analysis Type	Deg. Freedom	Sensitivity Coefficient	Component Uncertainty	Adjusted Mean
Temp_Effect_Output	0.00044 mV/deg F	95.45	0.00022 mV/deg F	B	inf	5.0	0.0011 V	0 mV/deg F
Temp_Effect_Zero	0.001 mV/deg F	95.45	0.0005 mV/deg F	B	inf	5.0	0.0025 V	0 mV/deg F
Temp_Range_DegF	2.0 deg F	99.00	0.78 deg F	B	inf			10
Cal_Weight	100 mg	99.00	38.8 mg	B	inf	0.004410	0.000171 V	1,000 g
Sensitivity						500.0		0.00882 mV/g
Load_Cell_Nonlinearity	0.01 mV	95.45	0.005 mV	B	inf	0.50	0.0025 V	0 mV
Load_Cell_Hysteresis	0.01 mV	95.45	0.005 mV	B	inf	0.50	0.0025 V	0 mV
Random			0.00153 mV	A	2	0.50	0.000765 V	0.0590 mV
Load_Cell_Noise	0.01 mV	95.45	0.005 mV	B	inf	0.50	0.0025 V	0 mV
Zero_Offset	0.2 mV	95.45	0.1 mV	B	inf	0.50	0.05 V	0 mV
Amplifier_Gain						8.879		0.5 V/mV
Gain_Acc	5 mV	95.45	2.5 mV	B	inf	1.0	0.0025 V	0 V
Amplifier_Stability	1.0 mV	95.45	0.5 mV	B	inf	1.0	0.0005 V	0 V
Amplifier_Nonlinearity	1 mV	95.45	0.5 mV	B	inf	1.0	0.0005 V	0 V
Amplifier_Noise	3 mV	99.00	1.165 mV	B	inf	1.0	0.001165 V	0 V
Bal_Stability	20 mV	95.45	10.0 mV	B	inf	1.0	0.01 V	0 V
Temp_Coeff	2 mV/deg C	95.45	1.0 mV/deg C	B	inf	5.60	0.0056 V	0 V/deg C
Temp_Range_DegC	1.1 deg C	99.00	0.43 deg C	B	inf			5.6
DMM_Accuracy	10.47 mV	95.45	5.23 mV	B	inf	1.0	0.00523 V	0 V
DMM_Resolution	0.5 mV	100.00	0.29 mV	B	inf	1.0	0.00029 V	0 V

Analysis Summary:
Computed Parameter Value: 4.44 V
Estimated Uncertainty: 51.9 mV
Distribution: Normal
Degrees of Freedom: infinite

Figure 9-7. Multivariate Analysis Report for Load Cell Calibration System.

9.3.3 <u>Wingboom Angle of Attack</u>. In this example, the uncertainty in the wingboom angle of attack (AOA) measurement is estimated using a system model analysis approach. Results obtained from Excel spreadsheet analysis are compared to the analysis results obtained with UncertaintyAnalyzer.

The Wingboom AOA measurement system is shown schematically in the Figure 9-8. Wingboom AOA is measured with a Model 1201 5k Ohm potentiometer manufactured by BEI Technologies, Inc. The potentiometer output voltage is run through a CDAU SCD-108S signal conditioning card (SCC). The SCC outputs a range of 0 to 4095 counts, limited to positive and negative potentiometer voltages that correspond to angles ranging from -45° to $+45^o$. The counts output from the SCC are converted to wingboom angle using the linear regression equation obtained from calibration data.
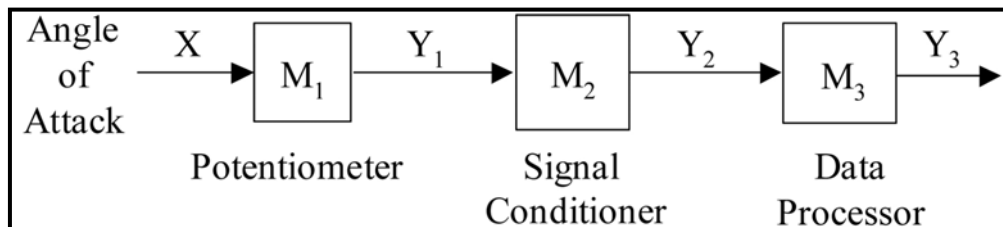
Figure 9-8. Block Diagram of Wingboom AOA Measurement System.

The uncertainty in the Wingboom AOA is estimated using a system model analysis approach. Results obtained from Excel spreadsheet analysis are compared to similar analysis using UncertaintyAnalyzer. Analysis details are presented in Appendix C.

9.3.3.1 <u>System Model Results</u>. Comparisons of the computed Potentiometer Module output, total uncertainty and degrees of freedom obtained from Excel spreadsheet and UncertaintyAnalyzer are listed in Table 9-11.

Table 9-11. Comparison of Potentiometer Total Uncertainty and Degrees of Freedom

	Excel Spreadsheet	UncertaintyAnalyzer
Potentiometer Output	0.282 V	0.282 V
Std Uncertainty	5.53 mV	5.53 mV
Deg. Of Freedom	Infinite	Infinite

The estimated uncertainties for the Potentiometer Module obtained from Excel spreadsheet calculations are listed in Table 9-12. The UncertaintyAnalyzer output report for the Load Cell Module is shown in Figure 9-9.

Table 9-12. Spreadsheet Analysis Results for Potentiometer Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Supply_Voltage	0.015 V	95	0.00765 V	0.0565	0.0004 V	5 V
Fixture_Angle	0.25 deg	99	0.0971 deg	0.01412	0.0014 V	20 deg
Max_Angle	2 deg	95	1.02 deg	-0.0008	-0.0008 V	354 deg
Nom_Sensitivity						1 V/V/deg
Linearity	0.005 V/V/deg	95	0.0026 V/V/deg	0.282	0.00072 V	0 V/V/deg
Resistance	0.03 V/V/deg	95	0.0153 V/V/deg	0.282	0.0043 V	0 V/V/deg
Noise	0.02 V/V/deg	95	0.0102 V/V/deg	0.282	0.0029 V	0 V/V/deg
Resolution	0.0011 V/V/deg	100	0.000635 V/V/deg	0.282	0.000179 V	0 V/V/deg
Temp_Coeff	$7e^{-5}$ V/V/deg	95	$3.57e^{-5}$ V/V/deg	14.1	0.0005 V	0 mV/deg F
Temp_Range	2 ^{o}C	99	1.02 ^{o}C			50 ^{o}C

where

Supply_Voltage	= Excitation Voltage
Nom_Sensitivity	= Potentiometer Sensitivity
Linearity	= Potentiometer Nonlinearity
Resistance	= Potentiometer Resistance Error
Noise	= Noise
Resolution	= Resolution Error
Temp_Coeff	= Temperature Correction Error
Temp_Range	= Temperature Range

Uncertainty Analysis Report | System Module Analysis Results

Wingboom AOA Measurement System
26-Sep-2005

Module 1: Potentiometer
Input Measurement Area: Angle
Output Measurement Area: Voltage

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Excitation Voltage	5	0.015000	95.00	0.00765	B	inf	0.0565	0.00043
Maximum Potentiometer Angle	354	2.000000	95.00	1.02		inf	-0.00080	0.00
Repeat Measurements Error	0		95.00				1.0	
Potentiometer Sensitivity	1		95.00				0.282	
Sensitivity Linearity	0	0.00500	95.00	0.00255	B	inf	0.282	0.00072
Resistance Error	0	0.03000	95.00	0.0153	B	inf	0.282	0.0043
Noise	0	0.02000	95.00	0.0102	B	inf	0.282	0.0029
Resolution Error	0	0.0011000	100.00	0.000635	B	inf	0.282	0.000179
Temperature Coefficient	0	0.0000700	95.00	0.0000357	B	inf	14.1	0.0005044
Temperature Range	50	2.0000	95.00	1.02		inf		

Module Analysis Summary:
Module Input: 20 deg
Module Output: 0.282 V
Output Uncertainty: 5.53 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure 9-9. Potentiometer Module Report.

Comparisons of the computed Signal Conditioner Module output, total uncertainty and degrees of freedom obtained from Excel spreadsheet and UncertaintyAnalyzer are listed in Table 9-13.

Table 9-13. Comparison of Signal Conditioner Total Uncertainty and Degrees of Freedom		
	Excel Spreadsheet	**UncertaintyAnalyzer**
SC Output	2958.0 Counts	2958.0 Counts
Std Uncertainty	36.0 Counts	36.0 Counts
Deg. Of Freedom	Infinite	Infinite

Comparison of the estimated uncertainties for the Signal Conditioner Module, obtained from Excel spreadsheet calculations are listed in Table 9-14. The UncertaintyAnalyzer output report for the Signal Conditioner Module is shown in Figure 9-10.

Table 9-14. Spreadsheet Analysis Results for Signal Conditioner Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Pot_Output			0.00533 V	3221.4	0.0004 Counts	0.282 V
Accuracy	0.025 deg	99	0.0097 V	3221.4	0.0014 Counts	0 V
Quantization	0.6 V	100	0.346 V	3221.4	0.000179 Counts	0 V
Coeff1				0.282		3221.4 Counts/V
Coeff2				1		2048 Counts

where

Pot_Output	= Output from Potentiometer Module
Accuracy	= SCC Accuracy
Quantization	= Quantization
Coeff1	= Counts Conversion Coefficient
Coeff2	= Counts Conversion Coefficient

Uncertainty Analysis Report   System Module Analysis Results

Wingboom AOA Measurement System
26-Sep-2005

Module 2: Signal Conditioner
Input Parameter: Potentiometer
Input Measurement Area: Voltage
Output Measurement Area: A/D

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
SCD-108S Accuracy	0	0.0250	95.00	0.0097	B	inf	3,221.4	31.2656
Quantization Error	0	0.0006000	100.00	0.0003464	B	inf	3,221.4	1.1159257
Conversion Coefficient	3,221.4		95.00				0.282	
Conversion Coefficient	2,048		95.00				1.0	

Module Analysis Summary:
Module Input: 0.282 V
Input Uncertainty: 5.53 mV
Degrees of Freedom: Infinite
Module Output: 2,958.0 Counts
Output Uncertainty: 36 Counts
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure 9-10. Signal Conditioner Module Report.

Comparisons of the computed Data Processor Module output, total uncertainty and degrees of freedom obtained from Excel spreadsheet and UncertaintyAnalyzer are listed in Table 9-15. These values also represent to overall system output, total system uncertainty and degrees of freedom.

Table 9-15. Comparison of Data Processor Total Uncertainty and Degrees of Freedom		
	Excel Spreadsheet	**UncertaintyAnalyzer**
Data Processor Output	21.61 deg	21.61 deg
Std Uncertainty	0.917 deg	0.917deg
Deg. Of Freedom	Infinite	Infinite

The estimated uncertainties for the Data Processor Module obtained from Excel spreadsheet calculations are listed in Table 9-16. The UncertaintyAnalyzer output report for the Data Processor Module is shown in Figure 9-11. The UncertaintyAnalyzer overall system analysis report is shown in Figure 9-12.

Table 9-16. Spreadsheet Analysis Results for Data Processor Module						
Parameter Name	**± Error Limits**	**% Confidence**	**Standard Uncertainty**	**Sensitivity Coefficient**	**Component Uncertainty**	**Nominal or Mean Value**
SC_Output			36.0 Counts	0.0228	0.821 deg	2958.0 Counts
Coeff3				2958.0		0.0228 deg/Count
Coeff4				-1		45.83 deg
Regression_Error			0.408 deg	1	0.406 deg	0 deg

where

        SC_Output   = Output from Signal Conditioner Module

        Coeff3   = Regression Coefficient

        Coeff4   = Regression Coefficient

        Regression_Error   = Standard Error or Estimate

Uncertainty Analysis Report   **System Module Analysis Results**

Wingboom AOA Measurement System
26-Sep-2005

Module 3: Data Processor
Input Parameter: Signal Conditioner
Input Measurement Area: A/D
Output Measurement Area: Angle

Analysis Results:

Name	Value	± Error Limits	% Con- fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Regression Line Slope	0.0228		95.00				2,958.0	
Regression Line Intercept	45.83		95.00				-1.0	
Standard Error of Forecast	0	0.7997	95.00	0.408	B	inf	1.0	0.408

Module Analysis Summary:
Module Input: 2,958.0 Counts
Input Uncertainty: 36 Counts
Degrees of Freedom: Infinite
Module Output: 21.61 deg
Output Uncertainty: 0.917 deg
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type A

Figure 9-11. Data Processor Module Report.

Uncertainty Analysis Report   **System Analysis Results**

Wingboom AOA Measurement System
26-Sep-2005

System Name: Wingboom AOA Measurement System

Module Name	Input	Output	Uncertainty	Coefficient
Potentiometer	20 deg	0.282 V	5.53 mV	0.0
Signal Conditioner	0.282 V	2,958.0 Counts	36 Counts	0.0
Data Processor	2,958.0 Counts	21.61 deg	0.917 deg	1.000

System Analysis Summary:
System Input: 20 deg
System Output: 21.61 deg
System Uncertainty: 0.917 deg
Confidence Level: 95.00 %
Coverage Factor: 1.9629 deg
Degrees of Freedom: Infinite
Tolerance Limits: 1.8 deg

Figure 9-12. Wingboom AOA Measurement System Report.

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APPENDIX A

CYLINDER VOLUME ANALYSIS

1.0 General.

In this analysis example, a steel cylinder artifact with nominal design dimensions of 0.65 cm in length by 1.40 cm in diameter is with a micrometer. The objective is to estimate the uncertainty in the cylinder volume measurement.

The mathematical relationship between the cylinder volume in terms of length and diameter is given as

$$V = \pi L \left(\frac{D}{2} \right)^2$$

where L and D are the length and diameter components, respectively.

A comparison of the uncertainty analysis results from hand calculations and those obtained using UncertaintyAnalyzer are presented.

1.1 Error Model.

As discussed in Chapter 5, the partial derivative method gives us an expression for error in the cylinder volume, ε_V, in terms of the component errors ε_L and ε_D.

$$\varepsilon_V = c_L \varepsilon_L + c_D \varepsilon_D$$

where

$$c_L = \left(\frac{\partial V}{\partial L} \right) = \pi \left(\frac{D}{2} \right)^2$$

$$c_D = \left(\frac{\partial V}{\partial D} \right) = \pi L \left(\frac{D}{2} \right)$$

Axiom 3 and the **variance addition rule** gives us means of developing an equation for expressing the uncertainty in the cylinder volume.

$$u_V = \sqrt{\text{var}(\varepsilon_V)}$$
$$= \sqrt{\text{var}(c_L \varepsilon_L + c_D \varepsilon_D)}$$
$$= \sqrt{c_L^2 u_L^2 + c_D^2 u_D^2 + \rho_{LD} u_L u_D}$$

where ρ_{LD} is the correlation coefficient for the uncertainties in the length and diameter component errors.

The errors in the length and diameter components, ε_L and ε_D, can be expressed in terms of their constituent measurement process errors.

A-1

$$\varepsilon_L = \varepsilon_{Lbias} + \varepsilon_{Lran} + \varepsilon_{Lres} + \varepsilon_{Lop} + \varepsilon_{Lenv}$$

$$\varepsilon_D = \varepsilon_{Dbias} + \varepsilon_{Dran} + \varepsilon_{Dres} + \varepsilon_{Dop} + \varepsilon_{Denv}$$

Where

Bias (bias)	-	the bias in the micrometer readings.
Random (ran)	-	the error that produces different results from measurement to measurement.
Resolution (res)	-	the error due to the finite resolution of the micrometer readings.
Operator (op)	-	the error due to any systematic bias on the part of the measuring technician.
Environment (env)	-	the error in any thermal or other correction due to a departure from nominal conditions.

1.2 Uncertainty Estimates.

The uncertainty in each component is expressed in terms of the uncertainties in the error sources obtained using Axiom 3 and the variance addition rule. Operating on ε_L above with the variance operator gives, for the uncertainty in the length measurement,

$$u_L = \sqrt{u_{Lbias}^2 + u_{Lran}^2 + u_{Lres}^2 + u_{Loper}^2 + u_{Lenvir}^2}$$

Likewise, applying the variance operator to ε_D above gives, for the uncertainty in the diameter measurement,

$$u_D = \sqrt{u_{Dbias}^2 + u_{Dran}^2 + u_{Dres}^2 + u_{Doper}^2 + u_{Denvir}^2}$$

There are no terms correlating process uncertainties within each component expression because the length measurement process errors are independent of one another, as are the diameter measurement process errors.

However, some of the length measurement process errors may not be independent of some of the diameter measurement process errors. Therefore, we must consider possible cross-correlation terms. This will be addressed in section 9.3.1.2.

The methods of uncertainty estimation are

Bias (bias)	-	heuristically from tolerance limits and in-tolerance probabilities.
Random (ran)	-	statistically from a measurement sample.
Resolution (res)	-	heuristically from the measuring parameter resolution spec and assumptions about containment probability.
Operator (op)	-	heuristically as a function of measuring parameter resolution.
Environment (env)	-	heuristically from tolerances and in-tolerance probabilities for the environment monitoring equipment.

For this example, the nominal specifications for the steel cylinder are:

Length: 0.65 cm

Diameter: 1.40 cm

Volume: 1.0 cc

and the measurement process specifications are:

Measurement Bias:	± 0.1mm with 97.5% confidence
Digital Resolution:	± 0.1 mm
Ambient Temperature:	28 °C ± 0.5 °C with 95% confidence
Thermal Expansion Coefficient for Steel:	5.3 x 10-6 per °C
Thermal Expansion Coefficient for Micrometer:	1.2 x 10-6 per °C

Repeat measurements of the cylinder length and diameter yielded the data listed in Table A-1.

We will now use the above specifications to estimate the process error uncertainties and component uncertainties for the length and diameter measurements.

Table A-1. Offset from Nominal Values

Length Offset (mm)	Diameter Offset (mm)
0.4	0.2
0.3	0.3
0.3	0.4
0.4	0.5
0.5	0.3
0.3	0.2
0.4	0.4

1.2.1 <u>Length Measurement Uncertainties</u>. As previously discussed in Chapter 3, heuristic estimates are obtained from containment limits and a containment probability. Both the measurement bias and operator bias are considered to be normally distributed error sources. Therefore, the uncertainty in each error source is computed from the containment limits, ±L, and the inverse normal distribution function, $\Phi^{-1}(\cdot)$, computed from the containment probability, p.

$$u = \frac{L}{\Phi^{-1}\left[(1+p)/2\right]}$$

The inverse normal distribution function, can be found in statistics texts and in most spread-sheet programs. UncertaintyAnalyzer also computes this function, as well as probability density functions for other error distributions.

1.2.1.1 <u>Measurement Bias</u>. The uncertainty in the measuring parameter bias (i.e., the bias in the micrometer) can be expressed in terms of ± 0.1 mm containment limits with 97.5% containment probability.

$$u_{Lbias} = \frac{(0.01\,\text{cm})}{\Phi^{-1}\left[(1+0.975)/2\right]} = \frac{0.01\,\text{cm}}{2.2414} = 0.0030\,\text{cm}$$

1.2.1.2 <u>Operator Bias</u>. To estimate the uncertainty in the operator bias, we will assume containment limits that are based on roughly half of the resolution error with 90% containment probability.

$$u_{Lop} = \frac{(0.5)(0.01\,\text{cm})}{\Phi^{-1}\left[(1+0.90)/2\right]} = 0.0030\,\text{cm}$$

A-3

The containment limits for the operator bias are not necessarily based on resolution error. Best judgment and knowledge should be used in developing appropriate containment limits and confidence levels.

1.2.1.3 <u>Resolution Error</u>. To estimate the uncertainty in the resolution error, we will note that the micrometer has a digital readout. Therefore, the resolution error can be assumed to be uniformly distributed.

The uncertainty in resolution error can then be expressed in terms of ± 0.1 mm containment limits and 100% containment probability.

$$u_{Lres} = \frac{0.01\,\text{cm}}{\sqrt{3}} = 0.0029\,\text{cm}$$

1.2.1.4 <u>Environmental Factors Error</u>. For this error source, we are interested in determining the uncertainty in the length measurement due to thermal expansion affects. In this case, we must consider both the cylinder and micrometer expansion, as well as the uncertainty in the environmental temperature reading.

We will assume that the environmental temperature error is normally distributed. The uncertainty in the temperature measurement error is then expressed in terms of ± 0.5 °C containment limits and 95% containment probability.

$$u_{temp} = \frac{0.5}{\Phi^{-1}[(1+0.95)/2]} = 0.255\,^{\circ}\text{C}$$

The uncertainty in the environmental error resulting from thermal expansion affects is then expressed as

$$u_{Lenv} = [(5.3 - 1.2 \times 10^{-6} / \,^{\circ}\text{C})] \times (0.65\,\text{cm}) \times (0.255\,^{\circ}\text{C})$$
$$= 0.68 \times 10^{-6}\,\text{cm}$$

1.2.1.5 <u>Random Error</u>. The uncertainty in the random error in the length measurement is determined from the repeat measurements data. As previously discussed in Chapter 2, the uncertainty in the random error is equal to the standard deviation of the sample data. The standard deviation of the sample of length measurements is given by

$$s_L = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(L_i - \overline{L})^2}$$

where L_i is the ith reading and the mean value of the sample is computed as

$$\overline{L} = \frac{1}{n}(L_1 + L_2 + \ldots + L_n)$$

In this example, we have recorded the length measurements in offset units from the nominal length, L_0. The mean of the offset values for the length is given as

$$\bar{x} = \frac{(0.4+0.3+0.3+0.4+0.5+0.3+0.4)}{7}$$
$$= 0.37 \text{ mm or } 0.037 \text{ cm}$$

and the mean value of the length measurement is

$$\bar{L} = L_0 + \bar{x}$$
$$= (0.65+0.0037)\text{ cm}$$

The offset from nominal data gives us a standard deviation of

$$s_L = \sqrt{\frac{(0.03)^2+(0.07)^2+(0.07)^2+(0.03)^2+(0.13)^2+(0.07)^2+(0.03)^2}{6}}$$
$$= \sqrt{\frac{0.0027+0.0147+0.0169}{6}}$$
$$= 0.076 \text{ mm or } 0.0076 \text{ cm}$$

In this example, we will be reporting the nominal values, so we will need use the uncertainty in the mean value.

$$u_{Lran} = \frac{s_L}{\sqrt{n}} = \frac{0.0076}{\sqrt{7}} = 0.0029 \text{ cm}$$

1.2.2 Diameter Measurement Uncertainties. The methods for estimating the process errors for the diameter measurement uncertainties are the same as for the length measurement uncertainties.

1.2.2.1 Measurement Bias. Since the micrometer is used to measure both cylinder length and diameter, the uncertainties in the measurement bias errors are the same.

$$u_{Dbias} = u_{Lbias} = 0.0045 \text{ cm}$$

1.2.2.2 Operator Bias. The same operator is making the length and diameter measurements, so the uncertainties in the operator bias errors are the same.

$$u_{Dop} = u_{Lop} = 0.003 \text{ cm}$$

1.2.2.3 Resolution Error. The same is true of the uncertainties in the resolution error, since the micrometer is used for both the length and diameter measurements.

$$u_{Dres} = u_{Lres} = 0.0029\text{cm}$$

1.2.2.4 Environmental Factors Error. The only difference in the uncertainty due to environmental factors error is that we use the nominal diameter, instead of nominal length.

$$u_{Denv} = \left[\left(5.3 - 1.2 \times 10^{-6} \, /^{\circ}C\right)\right] \times (1.40 \, cm) \times \left(0.255 \, ^{\circ}C\right)$$
$$= 1.464 \times 10^{-6} \, cm$$

1.2.2.5 Random Error. The uncertainty in the random error in the length measurement is determined from the repeat measurements data. The mean of the offset values for the diameter is given as

$$\bar{x} = \frac{(0.2 + 0.3 + 0.4 + 0.5 + 0.3 + 0.2 + 0.4)}{7}$$
$$= 0.329 \, mm \text{ or } 0.0329 \, cm$$

and the mean value in the diameter measurement is

$$\bar{D} = D_0 + \bar{x}$$
$$= (1.40 + 0.033) \, cm$$

The offset from nominal data gives us a sample standard deviation of

$$s_D = \sqrt{\frac{(0.13)^2 + (0.03)^2 + (0.07)^2 + (0.17)^2 + (0.03)^2 + (0.13)^2 + (0.07)^2}{6}}$$
$$= \sqrt{\frac{0.0016 + 0.0009 + 0.0049 + 0.0289 + 0.0009 + 0.0169 + 0.0049}{6}}$$
$$= 0.11 \, mm \text{ or } 0.011 \, cm$$

and an uncertainty in the mean value of

$$u_{Dran} = \frac{s_D}{\sqrt{n}} = \frac{0.011}{\sqrt{7}} = 0.0042 \, cm$$

1.3 Error Source Correlations.

Before we combine uncertainties, we must consider if there are any possible cross-correlations between process uncertainties for the two components. First, let us review what we know about the cylinder measurement process.

1. Both length and diameter are measured using the same device (i.e., the micrometer).

2. All measurements are made by the same person (operator).

3. All measurements were made in the measuring environment.

Given this knowledge, we can assert that the following process uncertainties are cross-correlated between the length and diameter components:

- Measurement Bias – u_{Lbias} and u_{Dbias}

- Operator Bias – u_{Lop} and u_{Dop}

- Environmental Factors – u_{Lenv} and u_{Denv}

Second, we need to write an equation that expresses the correlation coefficient, ρ_{LD}, for the component uncertainties, u_L and u_D, in terms of the correlation coefficients of the process errors that are cross-correlated between components. The required expression is

$$\rho_{xy} = \frac{1}{u_x u_y} \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} \rho\left(\varepsilon_{xi}, \varepsilon_{yj}\right) u_{xi} u_{yj}$$

where u_L and u_D are the total component uncertainties and u_{Li} and u_{Dj} are the process uncertainties for the length and diameter components, respectively.

For the cylinder volume, we need only consider the cross-correlations between component measurement bias uncertainties, operator bias uncertainties, and environmental factors uncertainties. Let us consider the values for these three correlation coefficients.

1.3.1 Correlation between Component Measurement Biases. Since the same device is used to measure both length and diameter, the parameter bias for these measurements is the same. In this instance, the correlation coefficient, $\rho_{Lbias,Dbias}$, is equal to 1.0.

1.3.2 Correlation between Component Operator Biases. Although the same operator makes both measurements, human inconsistency prevents us from assigning a correlation coefficient equal to 1.0. However, we also know that the correlation coefficient should not be equal to zero either. Given that this is all we can say from heuristic considerations, we will set the correlation coefficient between length and diameter operator biases, $\rho_{Loper,Doper}$, equal to 0.5

1.3.3 Correlation between Component Environmental Factors Errors. Since the length and diameter measurements are made in the same environment, the correlation coefficient between the length and diameter environmental factors, $\rho_{Lenvir,Denvir}$, is also equal to 1.0.

1.4 Combined Uncertainties.

We can now expand the total uncertainty equation for cylinder volume, u_V, in terms of the process uncertainties.

$$\begin{aligned}
u_V^2 &= c_L^2 u_L^2 + c_D^2 u_D^2 + 2 c_L c_D \rho_{LD} u_L u_D \\
&= c_L^2 u_L^2 + c_D^2 u_D^2 \\
&\quad + 2 c_L c_D \left(\rho_{Lbias,Dbias} u_{Lbias} u_{Dbias} + \rho_{Loper,Doper} u_{Loper} u_{Doper} + \rho_{Lenvir,Denvir} u_{Lenvir} u_{Denvir} \right)
\end{aligned}$$

The component uncertainties, u_L and u_D, can be computed using the values for the process uncertainties obtained earlier.

$$\begin{aligned}
u_L &= \sqrt{u_{Lbias}^2 + u_{Lran}^2 + u_{Lres}^2 + u_{Loper}^2 + u_{Lenvir}^2} \\
&= \sqrt{(0.0045)^2 + (0.0029)^2 + (0.0029)^2 + (0.003)^2 + \left(0.68 \times 10^{-6}\right)^2} \\
&= 0.0068 \, \text{cm}
\end{aligned}$$

$$u_D = \sqrt{u_{Dbias}^2 + u_{Dran}^2 + u_{Dres}^2 + u_{Doper}^2 + u_{Denvir}^2}$$

$$= \sqrt{(0.0045)^2 + (0.0042)^2 + (0.0029)^2 + (0.003)^2 + (1.464 \times 10^{-6})^2}$$

$$= 0.0074 \, cm$$

We can now input the values for the correlation coefficients and express the total uncertainty as

$$u_V^2 = c_L^2 u_L^2 + c_D^2 u_D^2 + 2 c_L c_D \left(u_{Lbias} u_{Dbias} + 0.5 u_{Loper} u_{Doper} + u_{Linver} u_{Denvir} \right)$$

The coefficients c_L and c_D are computed from the partial derivatives system equation and using the mean values for L and D.

$$c_L = \pi \left(\frac{\overline{D}}{2} \right)^2 = 3.14159 \left(\frac{1.433}{2} \right)^2 = 1.613 \, cm^2$$

$$c_D = \pi \overline{L} \left(\frac{\overline{D}}{2} \right) = 3.14159 (0.687) \left(\frac{1.433}{2} \right) = 1.546 \, cm^2$$

Substituting the values for the process uncertainties, we get

$$u_V^2 = (1.613 \times 0.0068)^2 + (1.546 \times 0.0074)^2$$
$$+ 2(1.613)(1.546)(0.0045 \times 0.0045 + 0.5 \times 0.003 \times 0.003 \times 1.464 \times 10^{-12})$$

$$u_V^2 = 3.75 \times 10^{-4} \, cm^6$$

and the total uncertainty in the cylinder volume of

$$u_V = 0.0194 \, cm^3$$

1.5 Degrees of Freedom.

When uncertainties are combined, we need to know the degrees of freedom for the total uncertainty. We can compute the degrees of freedom for the uncertainty in the cylinder volume, u_V, from the Welch-Satterthwaite formula

$$v = \frac{u_{V^*}^4}{\dfrac{c_L^4 u_L^4}{v_L} + \dfrac{c_D^4 u_D^4}{v_D}}$$

where u_{V^*} is the total uncertainty computed without cross-correlations between component process uncertainties.

$$u_{V^*} = \sqrt{c_L^2 u_L^2 + c_D^2 u_D^2}$$
$$= \sqrt{(1.613 \times 0.0068)^2 + (1.546 \times 0.0074)^2}$$
$$= 0.0159 \, \text{cm}^3$$

The degrees of freedom for the component uncertainties, v_L and v_D are computed from

$$v_L = \frac{u_L^4}{\dfrac{u_{Lbias}^4}{v_{Lbias}} + \dfrac{u_{Lran}^4}{v_{Lran}} + \dfrac{u_{Lres}^4}{v_{Lres}} + \dfrac{u_{Loper}^4}{v_{Loper}} + \dfrac{u_{Lenvir}^4}{v_{Lenvir}}}$$

and

$$v_D = \frac{u_D^4}{\dfrac{u_{Dbias}^4}{v_{Dbias}} + \dfrac{u_{Dran}^4}{v_{Dran}} + \dfrac{u_{Dres}^4}{v_{Dres}} + \dfrac{u_{Doper}^4}{v_{Doper}} + \dfrac{u_{Denvir}^4}{v_{Denvir}}}$$

The degrees of freedom for all of the process uncertainties were assumed to be infinite, except for the uncertainty in the random error, u_{Lran} and u_{Dran}, which have degrees of freedom equal to 6 (i.e., sample size minus one). Therefore, the degrees of freedom for the component uncertainties can be computed from

$$v_L = v_{Lran} \frac{u_L^4}{u_{Lran}^4} = 6 \left(\frac{0.0068}{0.0029} \right)^4 = 181$$

$$v_D = v_{Dran} \frac{u_D^4}{u_{Dran}^4} = 6 \left(\frac{0.0074}{0.0042} \right)^4 = 58$$

where the degrees of are reported as whole numbers.

The degrees of freedom for the total uncertainty can then be computed from the above values.

$$v = \frac{u_{V^*}^4}{\dfrac{c_L^4 u_L^4}{v_L} + \dfrac{c_D^4 u_D^4}{v_D}} = \frac{(0.0159)^4}{\dfrac{(1.613 \times 0.0068)^4}{181} + \dfrac{(1.546 \times 0.0074)^4}{58}} = 170$$

1.6 Analysis Results

The cylinder volume is computed using the mean length and diameter.

$$V = \pi L \left(\frac{D}{2} \right)^2 = 3.15149 \left(\frac{1.433}{2} \right)^2 \text{cm}^3 = 1.108 \, \text{cm}^3$$

The standard uncertainties and component uncertainties for the measurement process errors are summarized in Table A-2. Component uncertainty is obtained by multiplying the standard uncertainty by the appropriate sensitivity coefficient.

Table A-2. Uncertainty Estimates for Cylinder Volume using Hand Calculations

Variable Name	Standard Uncertainty	± Error Limits	% Confidence	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
L0						0.65 cm
Lbias	0.0045 cm	0.1 mm	97.5	1.613	0.00726 cm3	0 cm
Lran	0.0029 cm			1.613	0.00467 cm3	0.037 cm
Lres	0.0029 cm	0.05 mm	100.0	1.613	0.00467 cm3	0 cm
Lop	0.0030 cm	0.05 mm	90.0	1.613	0.00484 cm3	0 cm
Lenv	0.00000068 cm			1.613	0.00000110 cm3	0 cm
D0						1.40 cm
Dbias	0.0045 cm	0.1 mm	97.5	1.546	0.00696 cm3	
Dran	0.0042 cm			1.546	0.00649 cm3	0.033 cm
Dres	0.0029 cm	0.05 mm	100.0	1.546	0.00448 cm3	0 cm
Dop	0.0030 cm	0.05 mm	90.0	1.546	0.00464 cm3	0 cm
Denv	0.00000146 cm			1.546	0.00000226 cm3	0 cm

where
- L0 = nominal cylinder length
- Lbias = measurement bias in length measurement
- Lran = length repeatability error
- Lres = length resolution error
- Lop = length operator bias
- Lenv = length environmental factors error
- D0 = nominal cylinder diameter
- Dbias = measurement bias in diameter measurement
- Dran = diameter repeatability error
- Dres = diameter resolution error
- Dop = diameter operator bias
- Denv = diameter environmental factors error

A similar analysis was conduced using the following UncertaintyAnalyzer program screens and worksheets:

- Multivariate Analysis Screen
- Error Source Worksheet
- Correlation Analysis Screen

The following equations were entered in the Multivariate Analysis Screen:

Volume = Pi * Length * (Diameter / 2) ^ 2

Length = L0 + Lbias + Lran + Lres + Lop + Lenv

Diameter = D0 + Dbias + Dran + Dres + Dop + Denv

The variables in the length and diameter equations were added to the Root Variables Data table of the Multivariate Analysis Screen. Nominal values and measurement process specifications for each variable were then entered in the corresponding Error Source Worksheets.

UncertaintyAnalyzer's Error Source Worksheets allow the user to specify the display precision of the uncertainty estimates. In this example, the display precision was set equal to that used for the hand calculations. The multivariate analysis report is shown in Figure A-1.

UncertaintyAnalyzer Cylinder Volume Analysis
14-Jul-2005

Subject Parameter Data:
Parameter Name: Cylinder Volume
Qualifier 1:
Qualifier 2:

Root Variables Analysis Results:

Variable Name	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg. Freedom	Sensitivity Coefficient	Component Uncertainty	Adjusted Mean
L0						1.6125		0.65 cm
Lbias	0.1 mm	97.50	0.045 mm	B	inf	1.6125	0.00726 cc	0 cm
Lran			0.029 mm	A	6	1.6125	0.00468 cc	0.0371 cm
Lres	0.05 mm	100.00	0.029 mm	B	inf	1.6125	0.00468 cc	0 cm
Lop	0.05 mm	90.00	0.030 mm	B	inf	1.6125	0.00484 cc	0 cm
Lenv	0.0000133 mm	95.00	0.0000068 mm	B	inf	1.6125	0.000001097 cc	-0.00000933 cm
D0						1.5465		1.40 cm
Dbias	0.1 mm	97.50	0.045 mm	B	inf	1.5465	0.00696 cc	0 cm
Dran			0.042 mm	A	6	1.5465	0.0065 cc	0.0329 cm
Dres	0.05 mm	100.00	0.029 mm	B	inf	1.5465	0.004485 cc	0 cm
Dop	0.05 mm	90.00	0.030 mm	B	inf	1.5465	0.00464 cc	0 cm
Denv	0.0000287 mm	95.00	0.0000146 mm	B	inf	1.5465	0.00000226 cc	-0.0000201 cm

Analysis Summary:
Computed Parameter Value: 1.1080 cc
Estimated Uncertainty: 0.0194 cc
Distribution: Student's t
Degrees of Freedom: 169

Figure A-1. Multivariate Analysis Report for Cylinder Volume Example

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APPENDIX B

LOAD CELL ANALYSIS

1.0 General

The uncertainty in the load cell output voltage is estimated using both a system model analysis approach and a more general multivariate analysis method. Results obtained from Excel spreadsheet analysis are compared to similar analyses using UnceratintyAnalyzer.

1.1 Measurement Process

In this example, a load cell (i.e., tension transducer) is calibrated using a weight standard, as illustrated in Figure B-1. The calibration weight is extended from the load cell via a monofilament line. Repeat measurements of DC voltage are obtained by adding and removing the calibration weight. The DC voltage output from an amplifier/signal conditioner is measured with a digital multimeter.
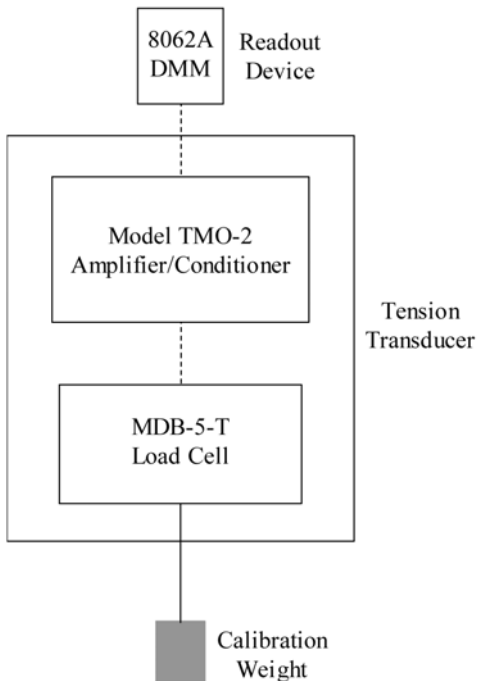

Figure B-1. Load Cell Calibration Setup

The purpose of this analysis is to estimate and report the total uncertainty in the average DC voltage obtained via the load cell calibration process. In this analysis, error in the mass of the calibration weight, errors intrinsic to the measurement equipment used, and other process errors are considered. A list of applicable error sources is given below.

- Bias in the value of the calibration weight
- Errors associated with the MDB-5-T Load Cell
- Errors associated with the Model TMO-2 Amplifier/Signal Conditioner
- Errors associated with the 8062A Digital Multimeter
- Error associated with the repeat measurements taken

Error limits for the set of calibration weights used to calibrate the load cell were assumed to be ± 100 mg with 99.0 % confidence.

Manufacturer's published specifications for the Load Cell, Amplifier/Signal Conditioner and Digital Multimeter are listed in Tables B-1, B-2 and B-3, respectively.

Table B-1. MDB-5-T Load Cell Specifications[1]

Specification	Value	Units
Rated Output (R.O.)	0 - 20	mV
Nonlinearity	0.05% of R.O.	mV
Hysteresis	0.05% of R.O.	mV
Noise (Nonrepeatability)	0.05% of R.O.	mV
Zero Balance	1.0% of R.O.	mV
Compensated Temp. Range	60 to 160	°F
Temperature Effect on Output	0.005% of Load/°F	g/°F
Temperature Effect on Zero	0.005% of R.O./°F	mV/°F

Table B-2. TMO-2 Amplifier/Signal Conditioner Specifications[2]

Specification	Value	Units
Output Voltage	0 - 10	V
Gain	0.5	V/mV
Nonlinearity	0.01%	mV
Accuracy	0.05% of Full Scale	mV
Noise and Ripple	< 3	mV
Balance Stability	0.2%	mV
Gain Stability	0.01%	mV
Temperature Coefficient	0.02% of F.S./°C	mV/°C

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[1] Specifications obtained from www.tloadcells.com/mdb-load-cell.cfm

[2] Specifications obtained from www.tloadcells.com/TMO-2.cfm

Table B-3. 8062A Digital Multimeter DC Voltage Specifications[3]

Specification	Value	Units
200 mV Range Resolution	0.01	mV
200 mV Range Accuracy	0.05% of Reading + 2 digits resolution	mV
2 V Range Resolution	0.1	mV
2 V Range Accuracy	0.05% of Reading + 2 digits resolution	mV
20 V Range Resolution	1	mV
20 V Range Accuracy	0.07% of Reading + 2 digits resolution	mV

To estimate the uncertainty due to random or repeatability error, we will use the measurements listed in Table B-4.

Table B-4. Load Cell Calibration Readings

Repeat Measurement	DC mV Measured	DC mV Offset from Nominal
1	8.866	0.056
2	8.872	0.061
3	8.870	0.060

1.2 System Model

We now need to define the mathematical relationship between the quantity being investigated and its component variables. In this case, measurement is made through a linear sequence of stages as shown in Figure B-2.

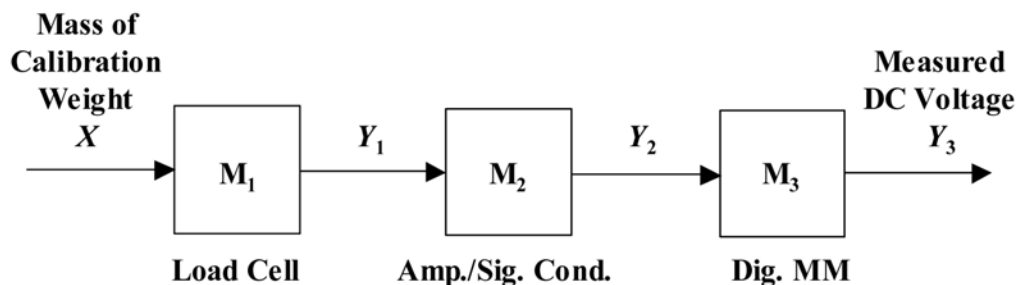**Figure B-2. Block Diagram of Load Cell Calibration System.**

The output, *Y*, from any given module of the system may comprise the input of another module or modules. Since each module's output carries with it an element of uncertainty, this means that this uncertainty may be present at the input of a subsequent module.

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[3] Specifications from 8062A Instruction Manual downloaded www.fluke.com

As one would expect, system uncertainty analysis follows a structured procedure. First, we must develop a measurement system model that describes the stages or modules involved in the processing the measurement of interest and identify the hardware and software used. We have already done this via Tables B-1 through B-4 and the system block diagram.

1.3 Module Output Equations

Second, we must develop a set of equations that describe module outputs in terms of inputs and identify the parameters that characterize these processes. From the module equations, we then identify and describe functions and parameters that may contribute to the error in the system output voltage.

1.3.1 <u>Load Cell Module (M_1)</u>. The first module consists of an MDB-5-T load cell manufactured by Transducer Techniques, Inc. This load cell is rated to output 0-20 mV for loads up to 5 lbs (2,270 g). According to manufacturer specifications listed in Table B-1, we need to consider the following error sources:

- Nonlinearity
- Hysteresis
- Repeatability
- Noise
- Zero Balance
- Temperature Effect on Output
- Temperature Effect on Zero

When developing an equation for the load cell output value Y_1, we must consider what impact these error sources will have on the measured value. We will briefly discuss each of the error sources list above and decide how they should be accounted for in the load cell output equation.

1.3.1.1 <u>Nonlinearity</u>. Nonlinearity is a measure of the deviation of the actual input-to-output performance of the device from an ideal linear relationship. Nonlinearity error is fixed at any given input, but varies with magnitude and sign over a range of inputs. Therefore, it is considered to be a bias error that is normally distributed. In this analysis, we will interpret the manufacturer specification of ± 0.05% of the rated output to be the 2-sigma limits or 95.45% confidence interval.

1.3.1.2 <u>Hysteresis</u>. Hysteresis indicates that the output of the device is dependent upon the direction and magnitude by which the input is changed. At any input value, hysteresis can be expressed as the difference between the ascending and descending outputs. Hysteresis error is fixed at any given input, but varies with magnitude and sign over a range of inputs. Therefore, it is considered to be a bias error that is normally distributed. In this analysis, we will interpret the manufacturer specification of ± 0.05% of the rated output to be the 2-sigma limits or 95.45% confidence interval.

1.3.1.3 <u>Repeatability</u>. Random error resulting from repeat measurements can result from various physical phenomena such as temperature variation or the act of removing and re-suspending the calibration weight multiple times. In this analysis, we will handle variation due to temperature

variation separately (see Thermal Effects). Uncertainty due to random error will be estimated using the data listed in Table B-4.

1.3.1.4 <u>Noise</u>. Nonrepeatability or random error intrinsic to the device, which causes the output to vary from observation to observation for a constant input is usually specified as noise. This error source varies with magnitude and sign over a range of inputs and is normally distributed. In this analysis, we will interpret the manufacturer specification of ±0.05% of the rated output to be the 2-sigma limits or 95.45% confidence level.

1.3.1.5 <u>Zero Balance</u>. Zero balance refers to the zero offset that occurs if the device exhibits a non-zero output for a zero input. Although zero offset error can be reduced by adjustment, there is no way to completely eliminate it because we do not know the true value of the offset. In this analysis, we will interpret the manufacturer specification of ±0.1% of the rated output to be the 2-sigma limits or 95.45% confidence level for a normally distributed error.

1.3.1.6 <u>Thermal Effects</u>. Temperature can affect both the offset and sensitivity of a device. To establish these effects, the device is tested at several temperatures within its operating range and the effect on zero and sensitivity or output are observed. The temperature effects errors specified by the manufacturer are assumed to be normally distributed and are interpreted to represent 2-sigma limits or 95.45% confidence interval. For this analysis, we will use a temperature range of 10 °F with error limits of ± 2 °F with an associated 99% confidence level. The temperature measurement error is also assumed to be normally distributed.

1.3.1.7 <u>Load Cell Output Equation</u>. In developing the equation to compute the load cell output as a function of the input load (i.e., calibration weight), we need to first assign a consistent naming convention for the various parameters. Based on our assessment of the load cell error sources, the relevant parameters are listed in Table B-5.

Table B-5. Parameters used in Load Cell Module Equation

Parameter Name	Description	Nominal or Mean Value	Error Limits	Percent Confidence
Cal_Weight	Calibration Weight or Load	1,000 g	± 100 mg	99.0
Sensitivity	Load Cell Sensitivity	0.00882 mV/g		
Nonlinearity	Nonlinearity	0	± 0.01 mV	95.45
Hysteresis	Hysteresis	0	± 0.01 mV	95.45
Noise	Nonrepeatability	0	± 0.01 mV	95.45
Zero_Offset	Zero Balance	0	± 0.2 mV	95.45
Temp_Range_degF	Temperature Range	10 °F	± 2.0 °F	99.0
Temp_Effect_Out	Temp Effect on Output	0	± 4.4 e-4 mV/°F	95.45
Temp_Effect_Zero	Temp Effect on Zero	0	± 0.001 mV/°F	95.45

The output equation for the load cell module is expressed as

Load_Cell_Output = Cal_Weight * Sensitivity + Nonlinearity + Hysteresis + Random + Noise + Zero_Offset + (Temp_Effect_Output + Temp_Effect_Zero) * Temp_Range_degF

1.3.2 <u>Amplifier Module (M$_2$)</u>. The TMO-2 Amplifier/Signal Conditioner, manufactured by Transducer Techniques Inc., amplifies the 0-20 mV signal from the load cell to a 0-10 V output signal. In this module, key error sources include:

- Gain Accuracy
- Gain Stability (or Instability)
- Nonlinearity
- Noise
- Balance Stability
- Temperature Correction

1.3.2.1 <u>Gain Accuracy</u>. Gain is the ratio of the amplifier output signal voltage to the input signal voltage. In this case, the TMO-2 amplifier has a nominal gain of 10V/20 mV or 0.5 V/mV. In this analysis, we will interpret the manufacturer specified accuracy of ± 0.05% of full scale to be the 2-sigma limits or the 95.45% confidence interval.

1.3.2.2 <u>Gain Stability</u>. If the amplifier voltage gain is represented by G_V, its input resistance by R and its feedback resistance by R_f, then oscillations are possible when

$$\frac{RG_V}{R+R_f} = \pi \; .$$

These oscillations appear as an instability in the amplifier gain. In this analysis we will interpret he manufacturer specification of 0.01% to be ± 0.01% of full scale. We will assume these limits to be 2-sigma limits or the 95.45% confidence interval.

1.3.2.3 <u>Nonlinearity</u>. As with the load cell module, actual amplifier response may depart from the ideal or assumed output versus input curve. Nonlinearity errors are point-by-point differences in actual versus expected response over the range of input signal levels. In this analysis, we will interpret the manufacturer specification of 0.01% to be ± 0.01% of full scale and that they are 2-sigma limits or the 95.45% confidence interval.

1.3.2.4 <u>Noise</u>. Noise generated within the amplifier that enters the signal path causes errors in the amplifier output. Since noise is directly related to gain, manufacturers usually specify noise error in absolute units of Volts RMS or Volts peak-to-peak. In this analysis, we will interpret the manufacturer specification of 3 mV peak-to-peak to be the 99% confidence interval.

1.3.2.5 <u>Balance Stability</u>. Balance stability, or instability, refers to a non-zero amplifier output exhibited for a zero input. Although balance instability can be reduced by adjustment, there is no way to completely eliminate it because we do not know the true value of the zero offset. In this analysis, we will interpret the manufacturer specification of ± 0.2% to be ± 0.2% of full scale and that this reflect the 2-sigma limits or the 95.45% confidence interval.

1.3.2.6 <u>Thermal Effects</u>. Both the balance (or zero) and gain are affected by temperature. Manufacturers generally state this as a temperature coefficient (Tempco) in terms of percent change or full scale per degree. In this analysis, we will interpret the manufacturer specification of ± 0.02% of full scale/°C to be the 2-sigma limits or the 95.45% confidence interval.

   To quantify the effect of temperature, however, we must establish the expected temperature change and use this with the temperature coefficient to compute expected variations. As with the

load cell module, we will estimate the impact of temperature correction error using a temperature range of 10 °F (5.6 °C) with measurement error limits of ± 1.1 °C with an associated confidence level of 99%.

1.3.2.7 <u>Amplifier Output Equation</u>. The output equation for the amplifier module is expressed below. Naming conventions and error limits for the various parameters are listed in Table B-6.

Amplifier_Output = Load_Cell_Output * Gain + Gain_Acc + Gain_Stability + Gain_Nonlinearity+ Gain_Noise + Bal_Stability + Temp_Coeff * Temp_Range_degC

Table B-6. Parameters used in Amplifier Module Equation

Parameter Name	Description	Nominal or Mean Value	Error Limits	Percent Confidence
Load_Cell_Output	Amplifier Input			
Gain	Gain	0.5 V/mV		
Gain_Acc	Gain Accuracy	0	± 5 mV	95.45
Gain_Stability	Gain Stability	0	± 1 mV	95.45
Gain_Nonlinearity	Nonlinearity	0 V	± 1 mV	95.45
Gain_Noise	Noise	0 V	± 3 mV	99.00
Bal_Stability	Balance Stability	0 V	± 20 mV	99.45
Temp_Coeff	Temperature Coefficient	0 V/ °C	± 2 mV/°C	95.45
Temp_Range_degC	Temperature Range	5.6 °C	± 1.1 °C	99.00

1.3.3 <u>Digital Multimeter Module (M_3)</u>. The 8602A Digital Multimeter, manufactured by Fluke, converts the Amplifier/Signal Conditioner module 0-10 V analog output signal to a digital signal and displays it on a readout. In this module, key error sources include:

- DC Volt Meter Accuracy
- DC Volt Meter Digital Resolution

1.3.3.1 <u>DC Voltage Accuracy</u>. The overall accuracy of the DC Voltage reading for a 0-20 V setting is specified as ± (0.07% of reading + 2 digits). In this analysis, we will interpret these specifications to be the 2-sigma limits or the 95.45% confidence level.

1.3.3.2 <u>Digital Resolution</u>. The digital resolution for the 0-20 V DC setting is specified as 1 mV. Since this is a digital display, the resolution error is uniformly distributed. Therefore, we will interpret the 100% confidence limits to be ± 0.5 mV.

1.3.3.3 Digital Multimeter Output Equation. The output equation for the digital multimeter module is expressed as

DMM_Output = Amplifier_Output + DMM_Accuracy + DMM_Resolution

The relevant equation parameters and error limits are listed in Table B-7.

Table B-7. Parameters used in Multimeter Module Equation

Parameter Name	Description	Nominal or Mean Value	Error Limits	Percent Confidence
Amplifier_Output	DMM Input			
DMM_Accuracy	DC Voltmeter Accuracy	0 V	± (0.07% Read +10 mV)	95.45
DMM_Resolution	DC Voltmeter Digital Resolution	0 V	± 0.5 mV	100

1.4 System Output and Total Uncertainty

The individual module equations and parameter information can now be used to estimate the output from the Load Cell Calibration System and the associated total uncertainty. There are a couple of ways to do this:

1. Analyze the three system modules separately and account for error propagation from module input to module output sequentially. This is the preferred analysis approach for this measurement scenario.

2. Alternatively, a multivariate analysis approach can be conducted using an overall system equation, and any associated nested variables equations, to describe the load cell calibration system.

If done correctly, the computed total system output and associated uncertainty should be the same for both analysis methods. These analysis methods will be described and compared using both Excel spreadsheet calculations and UncertaintyAnalyzer.

1.4.1 <u>System Analysis Method</u>. In the system analysis approach, each module is analyzed separately and the outputs and associated uncertainties for each module are propagated to subsequent modules. The appropriate module equations are repeated below for reference.

Load_Cell_Output = Cal_Weight * Sensitivity + Nonlinearity + Hysteresis + Random + Noise + Zero_Offset + (Temp_Effect_Output + Temp_Effect_Zero) * Temp_Range_degF

Amplifier_Output = Load_Cell_Output * Gain + Gain_Acc + Gain_Stability + Gain_Nonlinearity + Gain_Noise + Bal_Stability + Temp_Coeff * Temp_Range_degC

DMM_Output = Amplifier_Output + DMM_Accuracy + DMM_Resolution

When conducting this analysis via Excel spreadsheet, the partial derivative equations for each parameter and error source coefficient were determined offline. The partial derivative equations are listed below for reference.

Load Cell Sensitivity Coefficients

$$\frac{\partial Load\_Cell\_Output}{\partial Cal\_Weight} = Sensitivity \qquad \frac{\partial Load\_Cell\_Output}{\partial Sensitivity} = Cal\_Weight \qquad \frac{\partial Load\_Cell\_Output}{\partial Nonlinearity} = 1$$

$$\frac{\partial Load\_Cell\_Output}{\partial Hystersis} = 1 \qquad \frac{\partial Load\_Cell\_Output}{\partial Noise} = 1 \qquad \frac{\partial Load\_Cell\_Output}{\partial Zero\_Offset} = 1$$

$$\frac{\partial Load\_Cell\_Output}{\partial Temp\_Effect\_Output} = Temp\_Range\_degF \qquad \frac{\partial Load\_Cell\_Output}{\partial Temp\_Effect\_Zero} = Temp\_Range\_degF$$

$$\frac{\partial Load\_Cell\_Output}{\partial Temp\_Range\_degF} = Temp\_Effect\_Output + Temp\_Effect\_Zero$$

Amplifier Sensitivity Coefficients

$$\frac{\partial Amplifier\_Output}{\partial Gain} = Load\_Cell\_Output \qquad \frac{\partial Amplifier\_Output}{\partial Gain\_Acc} = 1 \qquad \frac{\partial Amplifier\_Output}{\partial Gain\_Stability} = 1$$

$$\frac{\partial Amplifier\_Output}{\partial Gain\_Nonlinearity} = 1 \qquad \frac{\partial Amplifier\_Output}{\partial Gain\_Noise} = 1 \qquad \frac{\partial Amplifier\_Output}{\partial Bal\_Stability} = 1$$

$$\frac{\partial Amplifier\_Output}{\partial Temp\_Coeff} = Temp\_Range\_degC \qquad \frac{\partial Amplifier\_Output}{\partial Temp\_Range\_degC} = Temp\_Coeff$$

Digital Multimeter Sensitivity Coefficients

$$\frac{\partial DMM\_Output}{\partial Amplifier\_Output} = 1 \qquad \frac{\partial DMM\_Output}{\partial DMM\_Accuracy} = 1 \qquad \frac{\partial DMM\_Output}{\partial DMM\_Resolution} = 1$$

The module equations, partial derivative equations and error source data were entered into an Excel spreadsheet. The inverse normal distribution function was also included to compute the coverage factors for specified confidence levels. The uncertainty analysis results for each module are summarized in Tables B-8 through B-10.

Table B-8 Spreadsheet Analysis Results for Load Cell Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Cal_Weight	0.1	99.00	0.0388 g	0.00882		1,000g
Sensitivity				1,000		0.00882 mV/g
Nonlinearity	0.01 mV	95.45	0.005 mV	1	0.005 mV	0 mV
Hysteresis	0.01 mV	95.45	0.005 mV	1	0.005 mV	0 mV
Noise	0.01 mV	95.45	0.005 mV	1	0.005 mV	0 mV
Random		95.45	0.0015 mV	1	0.0015 mV	0 mV
Zero_Offset	0.2 mV	95.45	0.1 mV	1	0.100 mV	0 mV
Temp_Effect_Out	0.00044 mV/F	95.45	0.00022 mV/F	0	0.00 mV	0 mV/deg F
Temp_Effect_Zero	0.001 mV/ F	95.45	0.0005 mV/F	0	0.00 mV	0 mV/deg F
Temp_Range_degF	2 F	99.00	0.776 F	0		10 deg F
Module Output	**8.88 mV**			**Tot. Unc. 0.1005 mV**		

Table B-9 Spreadsheet Analysis Results for Amplifier Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Load_Cell_Output			0.1005 mV			8.88 mV
Gain				8.88		0.5 V/mV
Gain_Acc	5 mV	95.45	2.5 mV	1	0.0025 V	0 V
Stability	1 mV	95.45	0.5 mV	1	0.0005 V	0 V
Nonlinearity	1 mV	95.45	0.5 mV	1	0.0005 V	0 V
Noise	3 mV	99.00	1.165	1	0.001165 V	0 V
Bal_Stability	20 mV	95.45	10 mV	1	0.01 V	0 V
Temp_Coeff	2 mV/C	95.45	1 mV/C	5.6	0.0056	0 mV/C
Temp_Range_degC	1.1 C	99.00			0 V	5.6 C
Module Output	**4.44 V**			**Tot. Unc. 51.6 mV**		

Table B-10 Spreadsheet Analysis Results for Digital Multimeter Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Amplifier_Output			51.6 mV	1	51.6 mV	4.44 V
DMM_Accuracy	10.47 mV	95.45	5.2 mV	1	5.2 mV	0 V
DMM_Resolution	0.5 mV	100	0.289 mV	1	0.289 mV	0 V
Module Output	**4.44 V**			**Tot. Unc. 51.9 mV**		

The overall output and total uncertainty for the Load Cell Calibration System is equal to the values computed for the last module in the series. Therefore, the overall output is 4.44 V with a total uncertainty of 51.9 mV.

A System Model analysis was conducted using UncertaintyAnalyzer by entering the module equations and associated information into the System Model screen. Partial derivatives were automatically computed, eliminating the time and potential error associated with doing this

manually. The overall system and module analysis reports are shown in Figures B-2 through B-5.

   With UncertaintyAnalyzer, the overall output and total uncertainty for the Load Cell Calibration System was also computed to be 4.44 V and 51.9 mV, respectively.

Uncertainty Analysis Report   **System Analysis Results**

UncertaintyAnalyzer System Model Analysis
14-Jul-2005

System Name: Load Cell Calibration System

Module Name	Input	Output	Uncertainty	Coefficient
Load Cell	1,000 g	8.88 mV	0.1005 mV	0
Amplifier	8.88 mV	4.44 V	51.6 mV	0
Digital Multimeter	4.44 V	4.44 V	51.9 mV	1.000

System Analysis Summary:
System Input: 1,000 g
System Output: 4.44 V
System Uncertainty: 51.9 mV
Confidence Level: 95.00 %
Coverage Factor: 1.9653 mV
Degrees of Freedom: Infinite
Tolerance Limits: 102 mV

Figure B-3. Load Cell Calibration System Report.

Uncertainty Analysis Report   **System Module Analysis Results**

UncertaintyAnalyzer System Model Analysis
Load Cell Calibration System

14-Jul-2005

Module 1: Load Cell
Input Measurement Area: Mass
Output Measurement Area: DC Voltage

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Load Cell Sensivity	0.00882		95.00				1,000.0	
Load Cell Nonlinearity	0	0.0100	95.45	0.005	B	inf	1.0	0.005
Load Cell Hysteresis	0	0.0100	95.45	0.005	B	inf	1.0	0.005
Error due to Repeat Measurements	0.0590	0.006523	95.00	0.001528	A	2	1.0	0.001528
Noise	0	0.0100	95.45	0.005	B	inf	1.0	0.005
Zero Balance	0	0.2000	95.45	0.1	B	inf	1.0	0.1
Temperature Effect on Output	0	0.00044	95.45	0.00022	B	inf	10.0	0.00220
Temperture Effect on Zero	0	0.0010	95.45	0.0005	B	inf	10.0	0.0050
Temperature Range	10	2.0000	99.00	0.776		inf		

Module Analysis Summary:
Module Input: 1,000 g
Module Output: 8.88 mV
Output Uncertainty: 0.1005 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type A,B

Figure B-4. Load Cell Module Report.

Uncertainty Analysis Report   **System Module Analysis Results**

UncertaintyAnalyzer System Model Analysis
Load Cell Calibration System

14-Jul-2005

Module 2: Amplifier
Input Parameter: Load Cell
Input Measurement Area: DC Voltage
Output Measurement Area: DC Voltage

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Amplifier Gain	0.5		95.00				8.88	
Amplifier Accuracy	0	0.0050	95.45	0.0025	B	inf	1.0	0.0025
Amplifier Stability	0	0.0010	95.45	0.0005	B	inf	1.0	0.0005
Amplifier Nonlinearity	0	0.0010	95.45	0.0005	B	inf	1.0	0.0005
Amplifier Noise	0	0.003000	99.00	0.001165	B	inf	1.0	0.001165
Balance Stability	0	0.0200	95.45	0.0100	B	inf	1.0	0.0100
Temperature Coefficient	0	0.0020	95.45	0.0010	B	inf	5.60	0.0056
Temperature Range	5.6	1.1000	99.00	0.427		inf		

Module Analysis Summary:
Module Input: 8.88 mV
Input Uncertainty: 0.1005 mV
Degrees of Freedom: Infinite
Module Output: 4.44 V
Output Uncertainty: 51.6 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure B-5. Amplifier Module Report

Uncertainty Analysis Report   **System Module Analysis Results**

UncertaintyAnalyzer System Model Analysis
Load Cell Calibration System

14-Jul-2005

Module 3: Digital Multimeter
Input Parameter: Amplifier
Input Measurement Area: DC Voltage
Output Measurement Area: DC Voltage

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Digital Multimeter Accuracy	0	0.01047	95.45	0.00523	B	inf	1.0	0.00523
Digitial Multimeter Resolution	0	0.0005	100.00	0.0002887	B	inf	1.0	0.0002887

Module Analysis Summary:
Module Input: 4.44 V
Input Uncertainty: 51.6 mV
Degrees of Freedom: Infinite
Module Output: 4.44 V
Output Uncertainty: 51.9 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure B-6. Digital Multimeter Module Report.

1.4.2 <u>Multivariate Analysis Method</u>. In the multivariate analysis approach, an overall equation is entered for the load cell calibration system, along with nested variables equations below.

Overall Load Cell Calibration System Equation

System_Output = Load_Cell_Output * Amplifier_Gain + Amplifier_Error + DMM_Error

Thermal Effects Equation

Temp_Drift = (Temp_Effect_Output + Temp_Effect_Zero) * Temp_Range_degF

Load Cell Output Equation

Load_Cell_Output = Cal_Weight * Sensitivity + Load _Cell_Nonlinearity + Load_Cell_Hysteresis + Random + Load_Cell_Noise + Zero_Offset + Temp_ Drift

Amplifier Output Equation

Amplifier_Error = Gain_Acc + Amplifier_Stability + Amplifier_Nonlinearity + Amplifier_Noise + Bal_Stability + Temp_Coeff * Temp_Range_DegC

Digital Multimeter Output Equation

DMM_Error = DMM_Accuracy + DMM_Resolution

The partial derivative equations are listed below. It is important to note that, we are only interested in the partial derivatives of the System Output equation with respect to the root variables or error sources, not the nested variables such as Temp_Drift, Load_Cell_Output, Amplifier_Error, or DMM_Error.

Sensitivity Coefficients

$$\frac{\partial System\_Output}{\partial Cal\_Weight} = Sensitivity * Amplifier\_Gain \qquad \frac{\partial System\_Output}{\partial Load\_Cell\_Nonlinearity} = Amplifier\_Gain$$

$$\frac{\partial System\_Output}{\partial Sensitivity} = Cal\_Weight * Amplifier\_Gain \qquad \frac{\partial System\_Output}{\partial Load\_Cell\_Hysteresis} = Amplifier\_Gain$$

$$\frac{\partial System\_Output}{\partial Random} = Amplifier\_Gain \qquad \frac{\partial System\_Output}{\partial Load\_Cell\_Noise} = Amplifier\_Gain$$

$$\frac{\partial System\_Output}{\partial Zero\_Offset} = Amplifier\_Gain \qquad \frac{\partial System\_Output}{\partial Temp\_Effect\_Output} = Temp\_Range * Amplifier\_Gain$$

$$\frac{\partial System\_Output}{\partial Temp\_Effect\_Zero} = Temp\_Range * Amplifier\_Gain$$

$$\frac{\partial System\_Output}{\partial Temp\_Range} = \left(Temp\_Effect\_Output + Temp\_Effect\_Zero\right) * Amplifier\_Gain$$

$$\frac{\partial System\_Output}{\partial Amplifier\_Gain} = Load\_Cell\_Output$$

$$\frac{\partial System\_Output}{\partial Gain\_Acc} = \frac{\partial System\_Output}{\partial Amplifier\_Error} * \frac{\partial Amplifier\_Error}{\partial Gain\_Acc} = 1$$

$$\frac{\partial System\_Output}{\partial Amplifier\_Stability} = \frac{\partial System\_Output}{\partial Amplifier\_Error} * \frac{\partial Amplifier\_Error}{\partial Amplifier\_Stability} = 1$$

$$\frac{\partial System\_Output}{\partial Amplifier\_Nonlinearity} = \frac{\partial System\_Output}{\partial Amplifier\_Error} * \frac{\partial Amplifier\_Error}{\partial Amplifier\_Nonlinearity} = 1$$

$$\frac{\partial System\_Output}{\partial Amplifier\_Noise} = \frac{\partial System\_Output}{\partial Amplifier\_Error} * \frac{\partial Amplifier\_Error}{\partial Amplifier\_Noise} = 1$$

$$\frac{\partial System\_Output}{\partial Bal\_Stability} = \frac{\partial System\_Output}{\partial Amplifier\_Error} * \frac{\partial Amplifier\_Error}{\partial Bal\_Stability} = 1$$

$$\frac{\partial System\_Output}{\partial Temp\_Coeff} = \frac{\partial System\_Output}{\partial Amplifier\_Error} * \frac{\partial Amplifier\_Error}{\partial Temp\_Coeff} = Temp\_Range\_DegC$$

$$\frac{\partial System\_Output}{\partial Temp\_Range\_DegC} = \frac{\partial System\_Output}{\partial Amplifier\_Error} * \frac{\partial Amplifier\_Error}{\partial Temp\_Range\_DegC} = Temp\_Coeff$$

$$\frac{\partial System\_Output}{\partial DMM\_Accuracy} = \frac{\partial System\_Output}{\partial DMM\_Error} * \frac{\partial DMM\_Error}{\partial DMM\_Accuracy} = 1$$

$$\frac{\partial System\_Output}{\partial DMM\_Resolution} = \frac{\partial System\_Output}{\partial DMM\_Error} * \frac{\partial DMM\_Error}{\partial DMM\_Resolution} = 1$$

The system output and nest variables equations, partial derivative equations and error source data were entered into an Excel spreadsheet. The inverse normal distribution function was also included to compute the coverage factors for specified confidence levels. The analysis results are summarized in Table B-11.

Table B-11. Multivariate Spreadsheet Analysis Results for Load Cell Calibration System

Variable Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Cal_Weight	0.1	99	0.0388 g	0.00441	0.000171 V	1,000g
Load_Cell_Nonlinearity	0.01 mV	95.45	0.005 mV	0.5	0.0025 V	0 mV
Sensitivity		95		500		0.00882 mV/g
Random			0.0015 mV	0.5	0.000764 V	0.0590 mV
Load_Cell_Hysteresis	0.01 mV	95.45	0.005 mV	0.5	0.0025 V	0 mV
Load_Cell_Noise	0.01 mV	95.45	0.005 mV	0.5	0.0025 mV	0 mV
Zero_Offset	0.2 mV	95.45	0.1 mV	0.5	0.05 V	0 mV
Temp_Effect_Output	0.00044 mV/F	95.45	0.00022 mV/F	5	0.0011 V	0 mV/deg F
Temp_Effect_Zero	0.001 mV/ F	95.45	0.0005 mV/F	5	0.0025 V	0 mV/deg F
Temp_Range_DegF	2 F	99	0.776 F			10 F
Amplifier_Gain		95		8.879		0.5 V/mV
Gain_Acc	5 mV	95.45	2.5 mV	1	0.0025 V	0 V
Amplifier_Stability	1 mV	95.45	0.5 mV	1	0.0005 V	0 V
Amplifier_Nonlinearity	1 mV	95.45	0.5 mV	1	0.0005 V	0 V
Amplifier_Noise	3 mV	99	1.165 mV	1	0.001165 V	0 V
Bal_Stability	20 mV	95.45	10 mV	1	0.01 V	0 V
Temp_Coeff	2 mV/C	95.45	1 mV/C	5.6	0.0056 V	0 V/deg C
Temp_Range_DegC	1 C	99	0.388 C			5.6 deg C
DMM_Accuracy	10.47 mV	95.45	5.23 mV	1	0.00523 V	0 V
DMM_Resolution	0.5 mV	100	0.29 mV	1	0.00029 V	0 V
System Output	**4.44 mV**			**Tot. Unc. 0.0519 V** **51.9 mV**		

A similar analysis was conducted using UncertaintyAnalyzer by entering the overall equation for the load cell calibration system, along with nested variables equations and associated information into the Multivariate Analysis screen. Partial derivatives were automatically computed by UncertaintyAnalyzer. The UncertaintyAnalyzer output report is shown in Figure B-6.

Uncertainty Analysis Report   **Multivariate Uncertainty Analysis**

UncertaintyAnalyzer Load Cell Calibration System Analysis
14-Jul-2005

Subject Parameter Data:
Parameter Name: Voltage Output from Load Cell Calibration System
Qualifier 1:
Qualifier 2:

Root Variables Analysis Results:

Variable Name	± Error Limits	% Confidence	Standard Uncertainty	Analysis Type	Deg. Freedom	Sensitivity Coefficient	Component Uncertainty	Adjusted Mean
Temp_Effect_Output	0.00044 mV/deg F	95.45	0.00022 mV/deg F	B	inf	5.0	0.0011 V	0 mV/deg F
Temp_Effect_Zero	0.001 mV/deg F	95.45	0.0005 mV/deg F	B	inf	5.0	0.0025 V	0 mV/deg F
Temp_Range_DegF	2.0 deg F	99.00	0.78 deg F	B	inf			10
Cal_Weight	100 mg	99.00	38.8 mg	B	inf	0.004410	0.000171 V	1,000 g
Sensitivity						500.0		0.00882 mV/g
Load_Cell_Nonlinearity	0.01 mV	95.45	0.005 mV	B	inf	0.50	0.0025 V	0 mV
Load_Cell_Hysteresis	0.01 mV	95.45	0.005 mV	B	inf	0.50	0.0025 V	0 mV
Random			0.00153 mV	A	2	0.50	0.000765 V	0.0590 mV
Load_Cell_Noise	0.01 mV	95.45	0.005 mV	B	inf	0.50	0.0025 V	0 mV
Zero_Offset	0.2 mV	95.45	0.1 mV	B	inf	0.50	0.05 V	0 mV
Amplifier_Gain						8.879		0.5 V/mV
Gain_Acc	5 mV	95.45	2.5 mV	B	inf	1.0	0.0025 V	0 V
Amplifier_Stability	1.0 mV	95.45	0.5 mV	B	inf	1.0	0.0005 V	0 V
Amplifier_Nonlinearity	1 mV	95.45	0.5 mV	B	inf	1.0	0.0005 V	0 V
Amplifier_Noise	3 mV	99.00	1.165 mV	B	inf	1.0	0.001165 V	0 V
Bal_Stability	20 mV	95.45	10.0 mV	B	inf	1.0	0.01 V	0 V
Temp_Coeff	2 mV/deg C	95.45	1.0 mV/deg C	B	inf	5.60	0.0056 V	0 V/deg C
Temp_Range_DegC	1.1 deg C	99.00	0.43 deg C	B	inf			5.6
DMM_Accuracy	10.47 mV	95.45	5.23 mV	B	inf	1.0	0.00523 V	0 V
DMM_Resolution	0.5 mV	100.00	0.29 mV	B	inf	1.0	0.00029 V	0 V

Analysis Summary:
Computed Parameter Value: 4.44 V
Estimated Uncertainty: 51.9 mV
Distribution: Normal
Degrees of Freedom: infinite
Analysis Category: Type A,B

Figure B-7. Multivariate Analysis Report for Load Cell Calibration System.

APPENDIX C

WINGBOOM AOA ANALYSIS

1.0 General

The uncertainty in the wingboom angle of attack (AOA) measurement is estimated using a system model analysis approach. Results obtained from Excel spreadsheet analysis are compared to the analysis results obtained with UncertaintyAnalyzer.

1.1 Measurement Process

The primary sensor for measuring wingboom angle of attack (AOA) is a BEI Model 1201 5k Ohm potentiometer with a maximum rotational travel of 354°. The potentiometer output voltage is run through a CDAU SCD-108S 8-channel signal conditioner, manufactured by Teletronics Technology Corporation, that consists of a ADG439F multiplexer, PGA204 amplifier, low-pass Butterworth filter, and analog to digital converter (ADC), as shown in the Figure C-1.
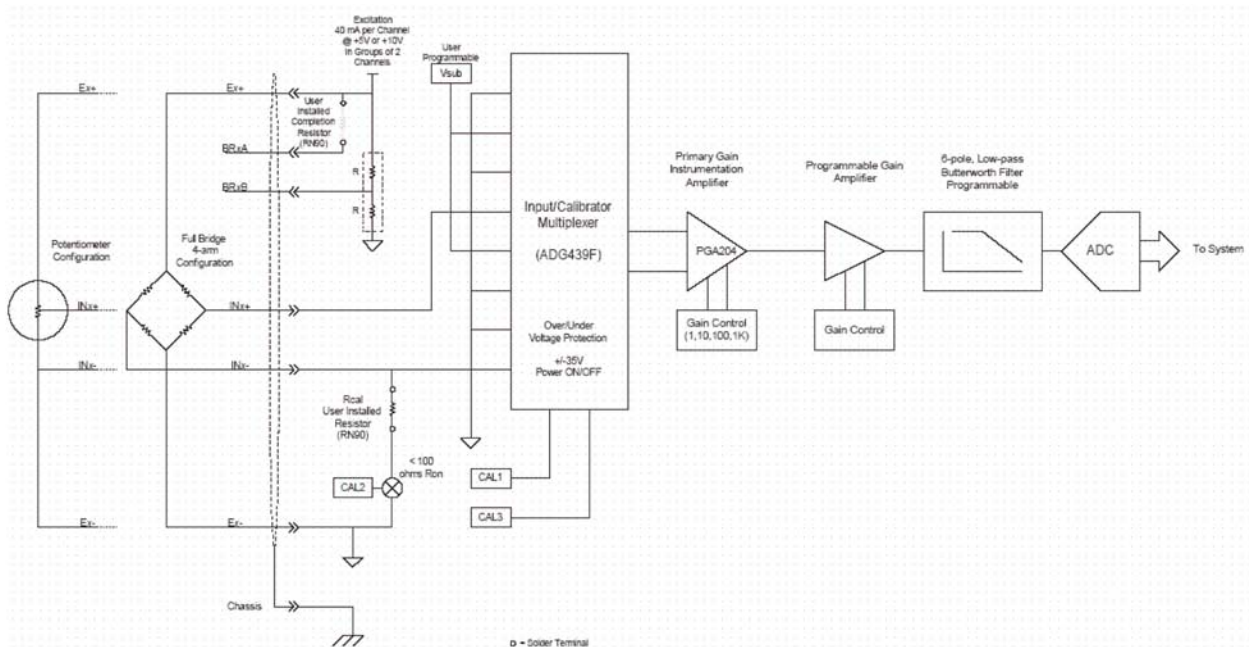
Figure C-1. Electrical Interface Diagram for SCD-108S Signal Conditioner[4]

The ADC uses 12-bit precision to convert the continuous voltage signal to a binary code. Therefore, the output signal from the ADC is a quantized value ranging from 0 to 4095 counts (i.e., 2^{12}-1).[5]

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[4] Figure 3 Electrical Interface Diagram for SCD-108S, Drawing No. 700004532-xxx Sheet 6.

[5] Email from Kenneth Miller, CIV NAVAIR to Dr. Howard Castrup, Integrated Sciences Group, Sent: 7/13/04 Subject: LSBF

The wingboom AOA measurement system is calibrated from - 45° to + 45° using an E-2C 535 Boom Universal Calibrator Fixture. The E-2C 535 Calibrator Fixture is, in turn, calibrated according to the LIST-A020 procedure.[6] The wingboom AOA calibration data are listed in Table C-1.

Table C-1 Wingboom AOA Calibration Data[7]

Meas. #	Counts	Angle	Meas. #	Counts	Angle
1	2052	0.875	18	2714	15.875
2	1878	-4.125	19	2931	20.875
3	1630	-9.125	20	3144	25.875
4	1378	-14.125	21	3361	30.875
5	1142	-19.125	22	3575	35.875
6	716	-29.125	23	3789	40.875
7	289	-39.125	24	4002	45.875
8	76	-44.125	25	3788	40.875
9	289	-39.125	26	3575	35.875
10	717	-29.125	27	3361	30.875
11	1142	-19.125	28	3144	25.875
12	1378	-14.125	29	2929	20.875
13	1629	-9.125	30	2712	15.875
14	1878	-4.125	31	2494	10.875
15	2052	0.875	32	2274	5.875
16	2273	5.875	33	2053	0.875
17	2496	10.875			

A regression analysis was conducted to obtain an unweighted least squares best fit (LSBF) to a straight line, as shown in Figure C-2. The straight line fit equation (1) is used to convert the recorded counts data to wingboom angle.

$$\text{Wingboom Angle} = 0.0228 * \text{Counts} - 45.83 \qquad \text{(C-1)}$$

Coefficient Significant Digits.

[6] Naval Air Test Center Technical Manual, Local Calibration Procedure LIST-A020, 1 November 2003.

[7] Calibration Data Sheet, C-2A, 162142 NP2000, TMATS File: H:\projects\C2 NP2000\C2np2k07.tma
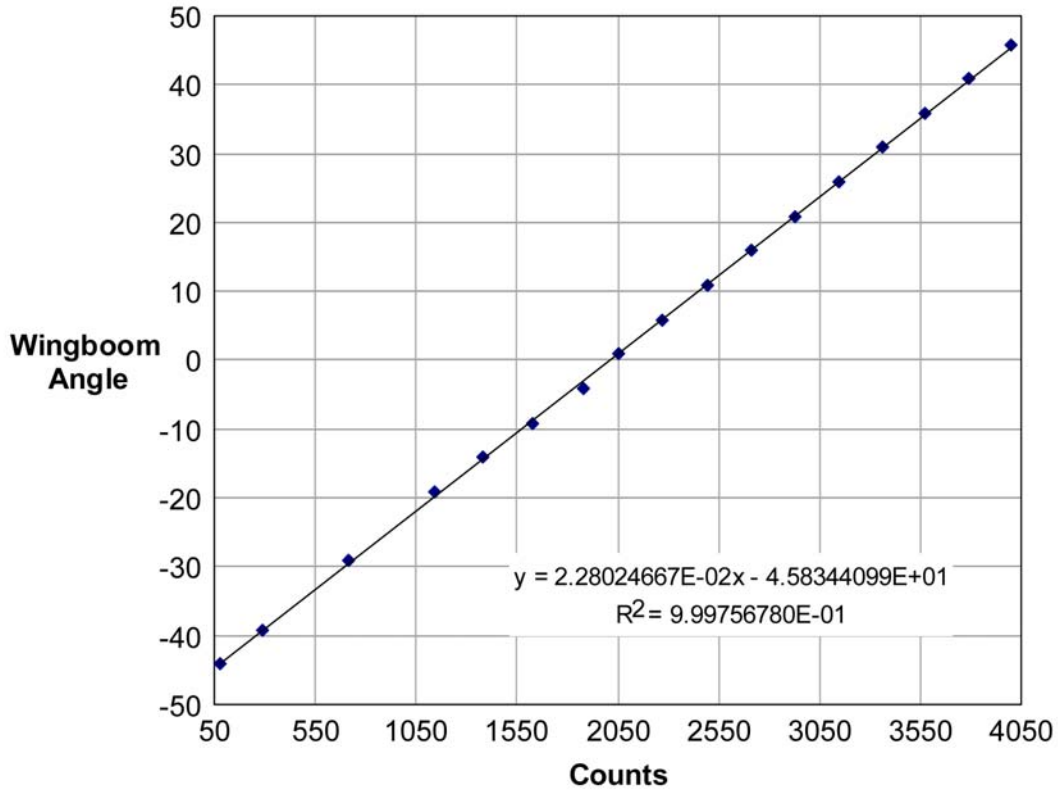
Figure C-2 Straight Line Fit of Calibration Data

1.2 System Model

The purpose of this analysis is to estimate the overall uncertainty in the wingboom AOA measurement. In this case, the wingboom AOA measurement is made through a linear sequences of stages or modules as shown in Figure C-3.
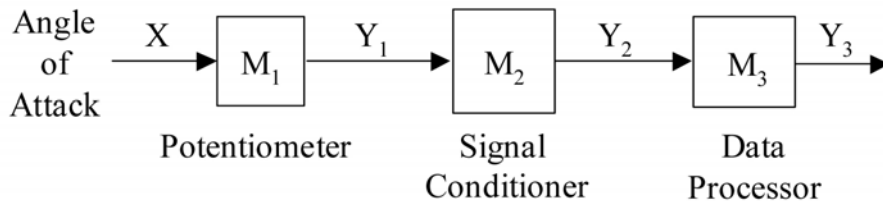

Figure C-3 Block Diagram of Wingboom AOA Measurement System

The output, *Y*, from any given system module comprises the input of the next module in the series. Since each module's output carries with it an element of uncertainty, this means that this uncertainty will be present at the input of a subsequent module.

As one would expect, system uncertainty analysis follows a structured procedure. We have already developed the requisite block diagram that describes the stages or modules involved in the measurement process. We now need to identify all relevant error sources and develop the mathematical equations that express the quantity of interest as a function of these errors and other key variables or parameters.

In this analysis, we will not consider errors due to signal attenuation or loss via module connectors, leads or interfaces. The error sources listed below are considered in this analysis:

- Bias in the E-2C 535 Boom Universal Calibrator Fixture.
- Error associated with the BEI Model 1201 5k Ohm potentiometer.
- Error associated with the CDAU SCD-108S signal conditioning card.
- Error associated with the LSBF straight line or calibration curve.

1.3 Module Output Equations

The following subsections describe the measurement system modules in detail, identifying error sources and defining appropriate module output equations. Manufacturer specifications will be used to establish error limits. Manufacturer specification documents, as well as other reference materials used in this analysis, are listed in the footnotes.

1.3.1 <u>Potentiometer Module (M_1)</u>. The first module consists of a Model 1201 5k Ohm potentiometer manufactured by BEI Technologies, Inc. Potentiometers are essentially a resistor, R_P, connected to a voltage source, V_I, with a moving contact or wiper.[8] The resistor is "divided" at the point of wiper contact and the voltage output signal, V_O, is proportional to the voltage drop across the resulting load resistance, R_L, as shown in Figure C-4a.

Most potentiometers are designed to generate a DC voltage output that is linearly proportional to rotational or lateral displacement X/X_P, as shown in Figure 4b. The potentiometer voltage output is then expressed as

$$V_o = V_I K \frac{X}{X_p}$$

where K is the potentiometer sensitivity, X is the sensed angle in degrees and Xp is the maximum angle that the potentiometer can travel.

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[8] Measurement, Instrumentation, and Sensors Handbook, CRCnetBase 1999, John G. Webster Editor-in Chief.
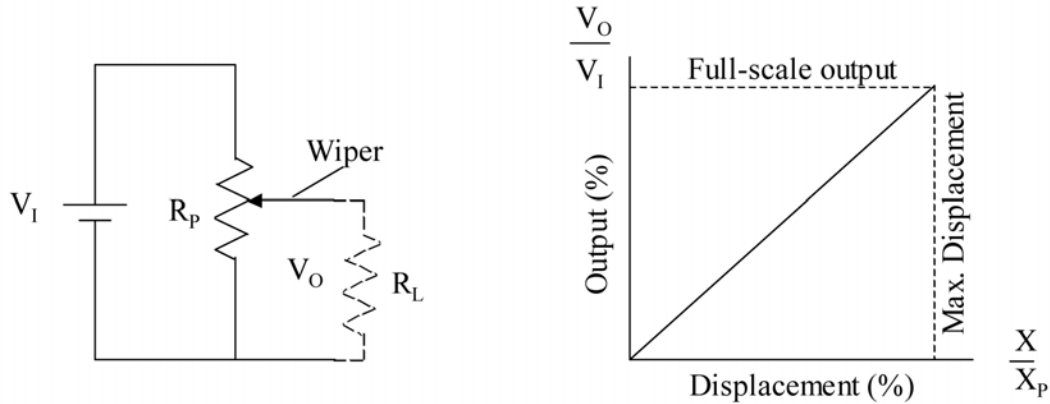
Figure C-4   a. Potentiometer Circuit   b. Ideal Linear Response Characteristic

When developing an uncertainty analysis for the potentiometer module, we must consider the impact that errors in V_I, K, X and X_P will have on the output value. In addition, manufacturer specifications for the Model 1201 5k Ohm potentiometer[9] indicate that there are other error sources:

- Sensed Angle (System Input)
- Supply Voltage
- Maximum Angle
- Sensitivity
- Linearity
- Resistance
- Noise
- Resolution
- Temperature Coefficient

Although hysteresis is not listed in the manufacturer specifications, we will attempt to assess its impact in this analysis via the LSBF regression error described in Module 3.

1.3.1.1 <u>System Input (X)</u>. In this analysis, a E-2C 535 Boom Universal Calibrator Fixture is used to provide a given wingboom angle. Therefore, any uncertainty in the angle established via the Calibrator Fixture must be determined and included in the analysis.

<u>1</u>.3.1.1.1 <u>Calibrator Fixture Bias</u>. The E-2C 535 calibrator fixture is, in turn, calibrated using the LIST-A020 procedure.[10] Accordingly, the Calibrator Fixture is reported to have tolerance limits of ± 0.25° of the angle established via the LIST-A020 procedure. For the purposes of this analysis, we will interpret these limits to be the 99% confidence interval and that the associated errors or biases are normally distributed. We will also use a nominal Calibrator Fixture angle of 20° for this analysis.

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[9] Specification sheet for BEI Model 1201 Servo Mount Wirewound Single-turn Precision Potentiometer, www.beiduncan.com.

[10] Naval Air Test Center Technical Manual, Local Calibration Procedure LIST-A020, 1 November 2003.

1.3.1.1.2 <u>Random Error</u>. Random error resulting from repeat measurements can result from variations in wingboom angle over a short time period or by testing or calibrating the potentiometer at a given angle multiple times.

Uncertainty due to random error can be estimated from test or calibration data at a nominal or mean wingboom angle. However, if the procedure used to calibrate the potentiometer includes the signal conditioner, then repeatability error should be evaluated in Module M_2.

1.3.1.2 <u>Supply Voltage (V_1)</u>. Since most sensors are passive, the signal conditioner must provide a regulated DC voltage or current via a precision power supply. According to the electrical interface diagram for the SCD-108S signal conditioner shown in Figure C-1, an external excitation voltage of 5 V or 10 V can be supplied to the potentiometer.

The excitation voltage accuracy is stated to be ± 0.3% maximum. In this analysis we assume that the error limits for the excitation voltage represent the 95% confidence interval and that the errors are normally distributed.

1.3.1.3 <u>Maximum Angle (X_P)</u>. The specification sheet for the BEI Model 1201 potentiometer indicates a maximum or actual electrical travel of 354° ± 2°. We interpret the ± 2° limits to represent a 95% confidence interval and assume that the associated errors are normally distributed.

1.3.1.4 <u>Sensitivity (K)</u>. The potentiometer sensitivity is essentially the dimensionless slope of the linear response curve shown in Figure C-4b. Ideally, the potentiometer sensitivity should have a value of unity. However, variation in potentiometer sensitivity can occur due to temperature effects, drift, hysteresis or other factors.

1.3.1.4.1 <u>Linearity</u>. Linearity, or more appropriately non-linearity, is a measure of the deviation of the actual input-to-output performance of the device from an ideal linear relationship. Linearity error is fixed at any given input, but varies with magnitude and sign over a range of inputs. Therefore, it is considered to be a bias error that is normally distributed.

The specification sheet for the BEI Model 1201 indicates linearity tolerance limits of ± 0.5% of full scale (FS) for standard conditions and ± 0.2% FS for best practical conditions. For the purposes of this analysis, we assume that the ± 0.5% FS limits apply. We also interpret these limits to represent a 95% confidence interval.

1.3.1.4.2 <u>Resistance</u>. Total resistance is a key parameter because it determines the amount of current drawn for a given applied voltage. Because potentiometer resistance can change over time between calibrations, it is important to estimate how resistance error impacts overall uncertainty.

Manufacturer specifications indicate that the resistance tolerance limits for the BEI Model 1201 are ± 3% FS for standard conditions and ± 1% FS for best practical conditions. For the purposes of this analysis, we assume that the ± 3% FS limits apply. We also interpret these limits to represent a 95% confidence interval and that the associated errors are normally distributed.

1.3.1.4.3 <u>Noise</u>. Non-repeatability or random error intrinsic to the device, that causes the output to vary from observation to observation for a constant input, is usually specified as noise.

Manufacturer specifications indicate that the equivalent noise resistance (ENR) has a maximum value of 100 Ohms. In this analysis, we interpret the specification of ± 100 Ohms to be a 95% confidence interval and the associated errors are normally distributed. Potentiometer noise, in relation to total potentiometer resistance of 5,000 Ohms, can be expressed as ± 2% of FS.

1.3.1.4.4 Resolution. Resolution defines the smallest possible increment of voltage change that can be produced and detected. In wire-wound coil potentiometers, resolution is the voltage drop in one turn of resistance wire. The best attainable resolution is $1/N \times 100\%$ of full scale voltage or resistance, where N is the number of turns in the coil. Resolution can also be expressed in terms of travel in inches or degrees.

The specification sheet for the BEI Model 1201 contains a footnote that indicates that Resolution Tables are available by model number and resistance value. The manufacturer was contacted and they indicated resolution limits of $\pm 0.11\%$ FS. We assume these limits to be the 95% confidence interval and that the associated errors are uniformly distributed.

1.3.1.4.5 Temperature Coefficient. Resistance increases with temperature. Therefore, the potentiometer sensitivity will be affected by temperature variation. However, this may not be a major concern as long as the changes in resistance are uniform and the potentiometer is operated within its rated temperature range. In general, wire-bound potentiometers have very low temperature coefficients.

The temperature coefficient tolerance limits specified for the BEI Model 1201 is $\pm 0.007\%/$ °C. In this analysis, we will assume these limits to be the 95% confidence interval and that the associated errors are normally distributed.

For the purposes of this analysis, we will assume a temperature range of 50°C with associated error limits of $\pm 2°$ C. We assume these limits to be the 95% confidence interval and that the associated errors are normally distributed.

1.3.1.5 Potentiometer Output Equation. The output equation for the potentiometer module is expressed in equation (C-3). Relevant information for the equation parameters is listed in Table C-2.

$$Pot\_Output = Supply\_Voltage * Sensitivity * Fixture\_Angle / Max\_Angle + Random \quad \text{(C-3)}$$
where

$$Sensitivity = Nom\_Sensitivity + Linearity + Resistance + Noise + Resolution + Temp\_Coeff * Temp\_Range$$

Table C-2 Parameters used in Potentiometer Module Equation

Parameter Name	Description	Nominal or Mean Value	Error Limits	Percent Confidence	Error
Supply_Voltage	Excitation Voltage	5 V	$\pm$ 15 mV	95.00	Normal
Fixture_Angle	Calibrator Fixture Angle	20 deg	$\pm$ 0.25 deg	99.00	Normal
Max_Angle	Maximum Potentiometer Angle	354 deg	$\pm$ 2 deg	95.00	Normal
Nom_Sensitivity	Potentiometer Sensitivity	1.0			
Linearity	Sensitivity Linearity	0	$\pm$ 0.005	95.00	Normal
Resistance	Resistance Error	0	$\pm$ 0.03	95.00	Normal
Noise	Noise	0	$\pm$ 0.02	95.00	Normal
Resolution	Resolution Error	0	$\pm$ 0.0011	100.00	Uniform
Temp_Coeff	Temperature Coefficient	0	$\pm 0.7e^{-4}/°C$	95.00	Normal
Temp_Range	Temperature Range	50 °C	$\pm$ 2 °C	95.00	Normal

1.3.2 Signal Conditioner Module (M_2). The SCD-108S signal conditioning card manufactured by Teletronics Technology Corporation (TTC). This card consists of an 8-channel multiplexer, amplifier, low-pass filter, and analog to digital converter. TTC specifies an overall system accuracy of $\pm\ 0.5\%$ for the SCD-108S card[11].

Correspondence with TTC clarified that the accuracy specification limits for the SCD-108S card is a percent full scale output and that the associated confidence level is 99%. In this analysis, we assume that overall errors associated with the signal conditioning card are normally distributed.

The voltage signal entering the SCD-108S card is converted to a quantized value ranging from 0 to 4095 counts (i.e., 2^{12}-1). During quantization, a finite number is used to represent a continuous value. The resulting resolution limit from the quantization of a 5 V signal using a 12-bit ADC is 5 V/(2^{12}) OR 1.2 mV. The quantization error limits are half the resolution or $\pm\ 0.6$ mV. In this case, the associated errors are uniformly distributed with a 100% confidence level.

1.3.2.1 Signal Conditioner Output Equation. The 0 to 4095 counts output range from SCD-108S card is limited to positive and negative voltages corresponding to angles ranging from - 45° to + 45°. The conversion from volts to counts is equal to 4095 counts/90° × 354°/5V or 3221.4 counts/V. In addition, 0° angle corresponds to 2^{11} or 2048 counts.

The output equation for the signal conditioner module is expressed in equation (C-4). Relevant information for the equation parameters are listed in Table C-3.

$$SC\_Output = (Pot\_Output + Accuracy + Quantization) * Coeff1 + Coeff2 \qquad \text{(C-4)}$$

Table C-3 Parameters used in Signal Conditioner Module Equation

Parameter Name	Description	Nominal or Mean Value	Error Limits	Percent Confidence	Error Distribution
Pot_Output	SCD-108S Input				
Accuracy	Signal Conditioner Accuracy	0 V	± 25 mV	99.00	Normal
Quantization	Quantization Error	0 V	± 0.6 mV	100.00	Uniform
Coeff1	Conversion Coefficient	3,221.4 Counts/V			
Coeff2	Conversion Coefficient	2048 Counts			

1.3.3 Data Processor Module (M_3). The data processing module takes the quantized AD converter counts output and computes a wingboom angle using the linear regression equation (C-1) obtained from calibration data. Errors associated with data processing result from computation round-off or truncation and from residual differences between an observed value during calibration and the value estimated from a regression equation.

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[11] SCD-108S Signal Conditioning Card Specifications, www.ttcdas.com.

In this analysis, we will focus on the error resulting from regression analysis of the calibration data, as shown in Figure C-2. The regression equation is typically expressed as

$$Y_R = b_0 + bX \tag{C-5}$$

where Y_R is the predicted value for a given X, b_0 is the value of Y when X equals zero, and b represents the amount of change in Y with X.

In regression analysis, the standard error of estimate is a measure of the difference between actual values and values estimated from a regression equation.[12] The standard error of estimate is also defined as the standard deviation of the normal distributions of Y for any given X.

1.3.3.1 <u>Standard Error of Estimate</u>. A regression analysis that has a small standard error has data points that are very close to the regression line. Conversely, a large standard error results when data points are widely dispersed around the regression line. The standard error of estimate is computed using equation (C-6).

$$s_{y,x} = \sqrt{\frac{\Sigma(Y - Y_R)^2}{n - 2}} \tag{C-6}$$

The calibration data listed in Table C-1 and the linear regression equation (C-1) were entered into a spreadsheet. The standard error of estimate was computed to be equal to 0.40° wingboom angle.

1.3.3.2 <u>Standard Error of Forecast</u>. As previously stated, the standard error is a measurement of the typical vertical distance of the sample data points from the regression line. However, we must also consider the fact that the regression line was generated from a finite sample of data. If another sample of data were collected, then a different regression line would result. Therefore, we must also consider the dispersion of various regression lines that would be generated from multiple sample sets around the true population regression line.

The standard error of the forecast accounts for the dispersion of the regression lines and is computed using equation (C-7).

$$s_f = s_{y,x}\sqrt{1 + \frac{1}{n} + \frac{(X - \bar{X})^2}{\Sigma((X - \bar{X})^2)}} \tag{C-7}$$

The standard error of forecast is computed for each value of X. For a wingboom angle of 20° in this analysis, which corresponds to a counts value of 2930, s_f has a value of 0.408° .

1.3.3.3 <u>Data Processor Output Equation</u>. The output equation for the data processing module is expressed in equation (C-8). Relevant information for the equation parameters are listed in Table (C-4).

$$Data\_Proc\_Output = Coeff3 * CS\_Output - Coeff4 + Regression\_Error \tag{C-8}$$

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[12] Hanke, J. et al.: *Statistatical Decision Models for Management*, Allyn and Bacon, Inc. 1984.

Table C-4. Parameters Used in Data Processor Module Equation

Parameter Name	Description	Nominal or Mean Value	Error Limits	Percent Confidence	Error Distribution
SC_Output	Data Processor Input				
Coeff3	Regression Line Slope	0.0228 deg/count			
Coeff4	Regression Line Intercept	45.83 deg			
Regression_Error	Standard Error of Forecast	0 deg.	0.408 deg.		

1.4 System Output and Total Uncertainty

In the system analysis approach, each module is analyzed separately and the output and associated uncertainties for each module are propagated to subsequent modules. The output for the Wingboom AOA Measurement System and the associated total uncertainty are equivalent to the output and uncertainty from the data processing module (M_3).

The three system modules were evaluated using UncertaintyAnalyzer's System Model screen and Excel spreadsheet calculations. If done correctly, the computed module outputs and associated uncertainties and the total system output and associated uncertainty should be the same for both analysis methods. The appropriate module equations are repeated below for reference.

$$Pot\_Output = Supply\_Voltage * Sensitivity * Fixture\_Angle / Max\_Angle + Random$$

$$Sensitivity = Nom\_Sensitivity + Linearity + Resistance + Noise + Resolution + Temp\_Coeff * Temp\_Rate$$

$$SC\_Output = (Pot\_Output + Accuracy + Quantization) * Coeff1 + Coeff2$$

$$Data\_Proc\_Output = Coeff3 * SC\_Output - Coeff4 + Regression\_Error$$

1.4.1 <u>Excel Spreadsheet Analysis</u>. When conducting this analysis via Excel spreadsheet, the partial derivative equations for each parameter and error source coefficient were determined offline. The partial derivative equations are listed below for reference.

Potentiometer Sensitivity Coefficients

$$\frac{\partial Pot\_Output}{\partial Supply\_Voltage} = Sensitivity * Fixture\_Angle / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Sensivity} = Supply\_Voltage * Fixture\_Angle / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Fixture\_Angle} = Supply\_Voltage * Sensitivity / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Max\_Angle} = -Supply\_Voltage * Sensitivity * Fixture\_Angle / Max\_Angle^2$$

$$\frac{\partial Pot\_Output}{\partial Nom\_Sensitivity} = \frac{\partial Pot\_Output}{\partial Sensitivity} \times \frac{\partial Sensitivity}{\partial Nom\_Sensitivity}$$
$$= Supply\_Voltage * Fixture\_Angle / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Linearity} = \frac{\partial Pot\_Output}{\partial Sensitivity} \times \frac{\partial Sensitivity}{\partial Linearity}$$
$$= Supply\_Voltage * Fixture\_Angle / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Resistance} = \frac{\partial Pot\_Output}{\partial Sensitivity} \times \frac{\partial Sensitivity}{\partial Resistance}$$
$$= Supply\_Voltage * Fixture\_Angle / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Noise} = \frac{\partial Pot\_Output}{\partial Sensitivity} \times \frac{\partial Sensitivity}{\partial Noise}$$
$$= Supply\_Voltage * Fixture\_Angle / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Resolution} = \frac{\partial Pot\_Output}{\partial Sensitivity} \times \frac{\partial Sensitivity}{\partial Resolution}$$
$$= Supply\_Voltage * Fixture\_Angle / Max\_Angle$$

$$\frac{\partial Pot\_Output}{\partial Temp\_Coeff} = \frac{\partial Pot\_Output}{\partial Sensitivity} \times \frac{\partial Sensitivity}{\partial Temp\_Coeff}$$
$$= (Supply\_Voltage * Fixture\_Angle / Max\_Angle) * Temp\_Range$$

$$\frac{\partial Pot\_Output}{\partial Temp\_Range} = \frac{\partial Pot\_Output}{\partial Sensitivity} \times \frac{\partial Sensitivity}{\partial Temp\_Range}$$
$$= (Supply\_Voltage * Fixture\_Angle / Max\_Angle) * Temp\_Coeff$$

Signal Conditioner Sensitivity Coefficients

$$\frac{\partial SC\_Output}{\partial Pot\_Output} = Coeff1$$

$$\frac{\partial SC\_Output}{\partial Accuracy} = Coeff1$$

$$\frac{\partial SC\_Output}{\partial Quantization} = Coeff1$$

$$\frac{\partial SC\_Output}{\partial Coeff1} = Pot\_Output + Accuracy + Quantization$$

$$\frac{\partial SC\_Output}{\partial Coeff2} = 1$$

Data Processor Sensitivity Coefficients

$$\frac{\partial Data\_Proc\_Output}{\partial SC\_Output} = Coeff3$$

$$\frac{\partial Data\_Proc\_Output}{\partial Coeff3} = SC\_Output$$

$$\frac{\partial Data\_Proc\_Output}{\partial Coeff4} = -1$$

$$\frac{\partial Data\_Proc\_Output}{\partial Regression\_Error} = 1$$

The module equations, partial derivative equations and error source data were entered into an Excel spreadsheet. The inverse normal distribution function was also included to compute the coverage factors for specified confidence levels. The uncertainty analysis results for each module are summarized in Tables C-5 through C-7.

Table C-5. Spreadsheet Analysis Results for Potentiometer Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Supply_Voltage	0.015 V	95	0.00765 V	0.0565	0.0004 V	5 V
Fixture_Angle	0.25 deg	99	0.0971 deg	0.01412	0.0014 V	20 deg
Max_Angle	2 deg	95	1.02 deg	-0.0008	-0.0008 V	354 deg
Nom_Sensitivity						1 V/V/deg
Linearity	0.005 V/V/deg	95	0.0026 V/V/deg	0.282	0.00072 V	0 V/V/deg
Resistance	0.03 V/V/deg	95	0.0153 V/V/deg	0.282	0.0043 V	0 V/V/deg
Noise	0.02 V/V/deg	95	0.0102 V/V/deg	0.282	0.0029 V	0 V/V/deg
Resolution	0.0011 V/V/deg	100	0.000635 V/V/deg	0.282	0.000179 V	0 V/V/deg
Temp_Coeff	$7e^{-5}$ V/V/deg/°C	95	$3.57e^{-5}$ V/V/deg/°C	14.1	0.0005 V	0 mV/deg F
Temp_Range	2 °C	99	1.02 °C			50 °C
Module Output	**0.282 V**			**Tot. Unc.**	**0.0553 V 5.53 mV**	

Table C-6. Spreadsheet Analysis Results for Signal Conditioner Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
Pot_Output			0.00553 V	3221.4	0.0004 Counts	0.282 V
Accuracy	0.025 V	99	0.0097 V	3221.4	0.0014 Counts	0 V
Quantization	0.06 V	100	0.346 V	3221.4	-0.000179 Counts	0 V
Coeff1				0.282		3221.4 Counts/V
Coeff2				1		2048 Counts
Module Output	**2958 Counts**			**Tot. Unc.**	**36.0 Counts**	

Table C-7. Spreadsheet Analysis Results for Data Processor Module

Parameter Name	± Error Limits	% Confidence	Standard Uncertainty	Sensitivity Coefficient	Component Uncertainty	Nominal or Mean Value
SC_Output			36.0 Counts	0.0228	0.821 deg	2958.0 Counts
Coeff1				2958.0		0.0228 deg/Count
Coeff2				-1		45.83 deg
Regression_Error			0.408 deg	1	.406 deg	0 deg
Module Output	**21.61 deg**			**Tot. Unc.**	**0.917 deg**	

The overall output and total uncertainty for the Wingboom AOA Measurement System is equal to the values computed for the last module in the series. Therefore, the overall output is 21.61 deg with a total uncertainty of 0.917 deg.

1.4.2 UncertaintyAnalyzer Results. A System Model analysis was conducted using Uncertainty Analyzer by entering the module equations and associated information into the System Model screen. Partial derivatives were automatically computed, eliminating the time and potential error associated with doing this manually. The overall system and module analysis reports are shown in Figures C-5 through C-8.

With UncertaintyAnalyzer, the overall output and total uncertainty for the Wingboom AOA Measurement System was also computed with (sic) to be 21.61 deg and 0.917 deg, respectively.

The table below is incorrect. It was copied/pasted twice. The correct table to be inserted as Figure C-5 can be found on the top of p. C-13 of the pdf document.

Uncertainty Analysis Report   **System Module Analysis Results**

Wingboom AOA Measurement System
26-Sep-2005

Module 1: Potentiometer
Input Measurement Area: Angle
Output Measurement Area: Voltage

Analysis Results:

Name	Value	± Error Limits	% Confidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Excitation Voltage	5	0.015000	95.00	0.00765	B	inf	0.0565	0.00043
Maximum Potentiometer Angle	354	2.000000	95.00	1.02		inf	-0.00080	0.00
Repeat Measurements Error	0		95.00				1.0	
Potentiometer Sensitivity	1		95.00				0.282	
Sensitivity Linearity	0	0.00500	95.00	0.00255	B	inf	0.282	0.00072
Resistance Error	0	0.03000	95.00	0.0153	B	inf	0.282	0.0043
Noise	0	0.02000	95.00	0.0102	B	inf	0.282	0.0029
Resolution Error	0	0.0011000	100.00	0.000635	B	inf	0.282	0.000179
Temperature Coefficient	0	0.0000700	95.00	0.0000357	B	inf	14.1	0.0005044
Temperature Range	50	2.0000	95.00	1.02		inf		

Module Analysis Summary:
Module Input: 20 deg
Module Output: 0.282 V
Output Uncertainty: 5.53 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure C-5. Wingboom AOA Measurement System Report.

Uncertainty Analysis Report   **System Module Analysis Results**

Wingboom AOA Measurement System
26-Sep-2005

Module 1: Potentiometer
Input Measurement Area: Angle
Output Measurement Area: Voltage

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Excitation Voltage	5	0.015000	95.00	0.00765	B	inf	0.0565	0.00043
Maximum Potentiometer Angle	354	2.000000	95.00	1.02		inf	-0.00080	0.00
Repeat Measurements Error	0		95.00				1.0	
Potentiometer Sensitivity	1		95.00				0.282	
Sensitivity Linearity	0	0.00500	95.00	0.00255	B	inf	0.282	0.00072
Resistance Error	0	0.03000	95.00	0.0153	B	inf	0.282	0.0043
Noise	0	0.02000	95.00	0.0102	B	inf	0.282	0.0029
Resolution Error	0	0.0011000	100.00	0.000635	B	inf	0.282	0.000179
Temperature Coefficient	0	0.0000700	95.00	0.0000357	B	inf	14.1	0.0005044
Temperature Range	50	2.0000	95.00	1.02		inf		

Module Analysis Summary:
Module Input: 20 deg
Module Output: 0.282 V
Output Uncertainty: 5.53 mV
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure C-6. Potentiometer Module Report.

Uncertainty Analysis Report   **System Module Analysis Results**

Wingboom AOA Measurement System
26-Sep-2005

Module 2: Signal Conditioner
Input Parameter: Potentiometer
Input Measurement Area: Voltage
Output Measurement Area: A/D

Analysis Results:

Name	Value	± Error Limits	% Con-fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
SCD-108S Accuracy	0	0.0250	95.00	0.0097	B	inf	3,221.4	31.2656
Quantization Error	0	0.0006000	100.00	0.0003464	B	inf	3,221.4	1.1159257
Conversion Coefficient	3,221.4		95.00				0.282	
Conversion Coefficient	2,048		95.00				1.0	

Module Analysis Summary:
Module Input: 0.282 V
Input Uncertainty: 5.53 mV
Degrees of Freedom: Infinite
Module Output: 2,958.0 Counts
Output Uncertainty: 36 Counts
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type B

Figure C-7. Signal Conditioner Module Report

Uncertainty Analysis Report **System Module Analysis Results**

Wingboom AOA Measurement System
26-Sep-2005

Module 3: Data Processor
Input Parameter: Signal Conditioner
Input Measurement Area: A/D
Output Measurement Area: Angle

Analysis Results:

Name	Value	± Error Limits	% Con- fidence	Standard Uncertainty	Analysis Type	Deg Freedom	Sensitivity Coefficient	Component Uncertainty
Regression Line Slope	0.0228		95.00				2,958.0	
Regression Line Intercept	45.83		95.00				-1.0	
Standard Error of Forecast	0	0.7997	95.00	0.408	B	inf	1.0	0.408

Module Analysis Summary:
Module Input: 2,958.0 Counts
Input Uncertainty: 36 Counts
Degrees of Freedom: Infinite
Module Output: 21.61 deg
Output Uncertainty: 0.917 deg
Distribution: Normal
Degrees of Freedom: Infinite
Analysis Category: Type A

Figure C-8. Data Processor Module Report

**** NOTHING FOLLOWS ****